Solutions to Mid II.

Q1 $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. [Union]

$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$. [CARTESIAN PRODUCT]

Let $(x,y) \in (A \times B) \cup (C \times D)$.

Then $(x,y) \in A \times B$ OR $(x,y) \in C \times D$.

$\Rightarrow$ $x \in A$ and $y \in B$ or $x \in C$ and $y \in D$

$\Rightarrow$ $x \in A \cup C$ and $y \in B \cup D$ or $x \in A \cup C$ and $y \in B \cup D$.

$\Rightarrow$ $x \in A \cup C$ and $y \in B \cup D$

$\Rightarrow$ $(x,y) \in (A \cup C) \times (B \cup D)$.

$$\boxed{(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)}$$

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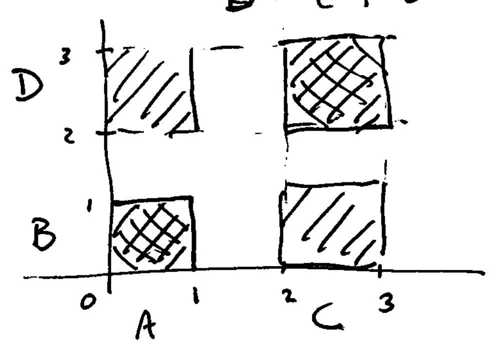
eg: Look at intervals in $\mathbb{R}$ $A = [0,1]$ $C = [2,3]$

$B = [0,1]$ $D = [2,3]$



$\Rightarrow$ extra in $(A \cup C) \times (B \cup D)$

$\Rightarrow$ $(A \times B) \cup (C \times D)$

Q2 $f : A \to B$ is <u>injective</u> if $\forall a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$.

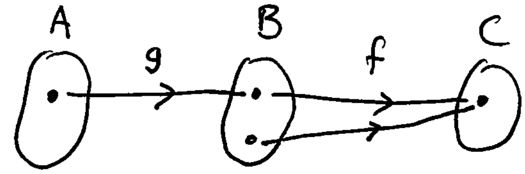
$f : A \to B$ is <u>surjective</u> if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

$f : A \to B$ is <u>bijective</u> if f is injective <u>and</u> f is surjective.

Given $f \circ g$ bijective :

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

1. No, f does not have to be injective.
   see example.



Example

2. Yes, f must be surjective.

   $\forall c \in C$, $\exists a \in A$ so that $(f \circ g)(a) = c$ --- since $f \circ g$ is bijective.

   $\Rightarrow f(g(a)) = c$. We've found $g(a) \in B$ so that $f(g(a)) = c$.

   $\Rightarrow f$ surjective.

3. Yes, g must be injective.

   $\forall a_1, a_2 \in A$, $g(a_1) = g(a_2) \Rightarrow f(g(a_1)) = f(g(a_2))$

   $\Rightarrow f \circ g(a_1) = f \circ g(a_2)$

   $\Rightarrow a_1 = a_2$ ---- since $f \circ g$ bijective

   $\Rightarrow g$ injective.

4. No, g does not have to be surjective.
   see example.

$\boxed{Q3}$ $f(S) = \{f(x) \mid x \in S\}$. [IMAGE] —— (i)

$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$. [PREIMAGE] —— (ii)

Let $y \in f(f^{-1}(T))$. By (i) this means $y = f(x)$ for some $x \in f^{-1}(T)$.

By (ii), $x \in f^{-1}(T)$ means $f(x) \in T$. $\Rightarrow y = f(x) \in T$

$\Rightarrow y \in T$.

$\Rightarrow \boxed{f(f^{-1}(T)) \subseteq T}$ —— (iii)

eg $f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$

$T = \{4, -13\}$

$f^{-1}(T) = \{\pm 2\}$

$f(f^{-1}(T)) = \{4\} \subsetneq \{4, -13\}$

Let $y \in T$. Given that f is <u>surjective</u>, there exists $x \in A$ so that $f(x) = y$. By (ii) $x \in f^{-1}(T)$.

By (i) $f(x) \in f\left(f^{-1}(T)\right)$

$\Rightarrow y \in f(f^{-1}(T))$.

& we've shown $T \subseteq f(f^{-1}(T))$ ... using f surj.

Combining with previous inclusion (iii), gives $T = f(f^{-1}(T))$.

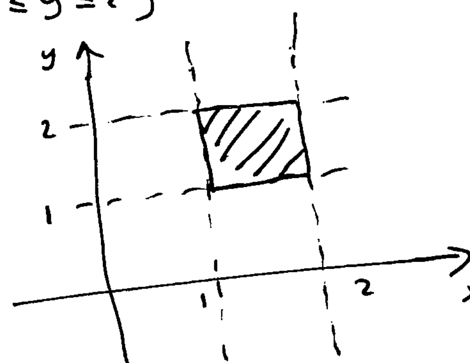
$\boxed{Q4}$ $\pi(1,1) = 1 = \pi(1,2)$ yet $(1,1) \neq (1,2)$. so π <u>not</u> <u>injective</u>.

$\forall x \in \mathbb{R}$, $x = \pi(x,0)$ & $(x,0) \in \mathbb{R}^2$ $\Rightarrow$ π <u>is</u> <u>surjective</u>.

$[1,2] \times [1,2] = \{(x,y) \mid x \in [1,2] \text{ and } y \in [1,2]\}$

$= \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 2,\ 1 \le y \le 2\}$

$=$ unit square shown



Thus $[1,2] \times [1,2] \subsetneq \pi^{-1}\left(\pi([1,2] \times [1,2])\right)$

$\pi\left([1,2] \times [1,2]\right) = \{\pi(x,y) \mid x \in [1,2],\ y \in [1,2]\}$

$= \{x \mid x \in [1,2],\ y \in [1,2]\}$

$= \{x \mid 1 \le x \le 2\} = [1,2]$

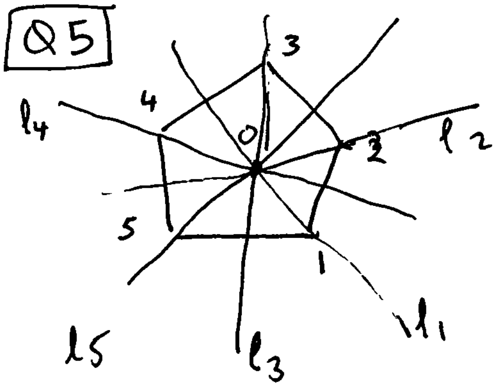
$\pi^{-1}\left(\pi\left([1,2] \times [1,2]\right)\right) = \pi^{-1}\left([1,2]\right)$

$= \{(x,y) \mid \pi(x,y) \in [1,2]\}$

$= \{(x,y) \mid x \in [1,2]\} = \{(x,y) \mid 1 \le x \le 2\}$

infinite vertical strip through $[1,2] \subseteq x$-axis

Q5



10 symmetries:

l_1, l_2, l_3, l_4, l_5 Reflections in lines

$\mathbb{1}, R, R^2, R^3, R^4$ Rotations

where $R = \frac{2\pi}{5}$ counterclockwise rotation about O.

$\mathrm{Symm}(\pentagon) \longrightarrow \mathrm{Perm}(\{1,2,3,4,5\})$

$f \longmapsto$ restriction of f to the set of vertices of pentagon $\{1,2,3,4,5\}$.

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in particular

$l_1 \longmapsto (25)(34)$	$\mathbb{1} \longmapsto \mathbb{1}$
$l_2 \longmapsto (13)(45)$	$R \longmapsto (12345)$
$l_3 \longmapsto (24)(15)$	$R^2 \longmapsto (13524)$
$l_4 \longmapsto (12)(35)$	$R^3 \longmapsto (14253)$
$l_5 \longmapsto (14)(23)$	$R^4 \longmapsto (15432)$

Q6

1. $\boxed{\text{FALSE}}$ $|\mathbb{P}(A)| = 2^{|A|}$

2. $\boxed{\text{TRUE}}$ $|A^A| = |A|^{|A|}$

3. $\boxed{\text{TRUE}}$ $\chi_{A\cap B} = \chi_A \cdot \chi_B$

4. $\boxed{\text{FALSE}}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$ ← de Morgan

5. $\boxed{\text{FALSE}}$ $(f\circ g)^{-1} = g^{-1}\circ f^{-1}$ ← (order switches)

6. $\boxed{\text{TRUE}}$ ... it is a special type of subset of $A\times B$

$\Rightarrow$ is a special type of element of $\mathbb{P}(A\times B)$.