<u>Prop</u> The map

$$\Psi : \mathcal{P}(A) \longrightarrow \{0,1\}^A$$
$$: S \longmapsto \chi_S$$

which takes $S \subseteq A$ (an element of the power set of A) to its <u>characteristic function</u> χ_S (defined by $\chi_S(a) = \begin{cases} 0 & a \notin S \\ 1 & a \in S \end{cases}$),

is a bijection.

<u>Proof</u> Just need to verify: Ψ is injective, Ψ is surjective.

(i) Ψ is injective:

$$\Psi(S_1) = \Psi(S_2) \Rightarrow \chi_{S_1} = \chi_{S_2} \quad \text{— (1)}$$

Thus

$$a \in S_1 \underset{\substack{\uparrow \\ \text{def}^{\text{n}} \text{ of} \\ \chi_{S_1}}}{\Longleftrightarrow} \chi_{S_1}(a) = 1 \underset{\substack{\uparrow \\ \text{by (1)}}}{\Longleftrightarrow} \chi_{S_2}(a) = 1 \underset{\substack{\uparrow \\ \text{def}^{\text{n}} \text{ of} \\ \chi_{S_2}}}{\Longleftrightarrow} a \in S_2$$

So $a \in S_1 \Leftrightarrow a \in S_2$; therefore $S_1 = S_2$ (def$^{\text{n}}$ of equality of sets).

$\Rightarrow \Psi$ injective.

(ii) Ψ is surjective: Given $f \in \{0,1\}^A$. This means f is a function, $f : A \to \{0,1\}$. Define a subset $S \subseteq A$ by

$S \overset{\text{def}}{=} \{a \in A \mid f(a) = 1\}$. By definition of S,

$a \in S \Leftrightarrow f(a) = 1$. But $a \in S \underset{\substack{\uparrow \\ \text{def}^{\text{n}} \text{ of char. function.}}}{\Leftrightarrow} \chi_S(a) = 1$

This means $f(a) = 1 \Leftrightarrow \chi_S(a) = 1$, or $\boxed{\chi_S = f}$

$\Rightarrow f = \chi_S = \Psi(S)$. $\Rightarrow \Psi$ surjective. $\blacksquare$

<u>Prop</u> The map $\Phi : B^{\{1,\dots,n\}} \longrightarrow B^n$

$: f \longmapsto (f(1), \dots, f(n))$

is a bijection.

<u>Proof</u> Need to verify Φ injective + Φ surjective.

(i) <u>Φ injective</u> Suppose $f_1, f_2 \in B^{\{1,\dots,n\}}$

and $\Phi(f_1) = \Phi(f_2)$.

This means $(f_1(1), \dots, f_1(n)) = (f_2(1), \dots, f_2(n))$.

By def^{n} of equality of ordered n-tuples, this means

$f_1(1) = f_2(1), \quad f_1(2) = f_2(2), \quad \dots, \quad f_1(n) = f_2(n)$.

Thus the functions f_1 and f_2 agree on <u>all</u> inputs $1, \dots, n$.

$\Rightarrow f_1 = f_2$. $\Rightarrow \Phi$ injective.

(ii) <u>Φ surjective</u>. Given <u>any</u> $(b_1, \dots, b_n) \in B^n$.

Define a function $f : \{1, \dots, n\} \to B$

by $f(1) = b_1, \dots, f(n) = b_n$.

But then $\Phi(f) = (f(1), \dots, f(n)) = (b_1, \dots, b_n)$

& so Φ is surjective.

$\blacksquare$