Prop: Let G_1 and G_2 be groups, and let $f: G_1 \to G_2$ respect the group multiplications:

$$\forall x_1, x_2 \in G_1 \qquad f(x_1 x_2) = f(x_1) f(x_2).$$

Then

$$(1) \quad f(e_1) = e_2 \qquad \text{where } e_i = \text{identity element of } G_i$$

and

$$(2) \quad f(x^{-1}) = f(x)^{-1} \qquad \forall x \in G_1.$$

Proof:

Proof of (1) Since e_1 is the identity element in G_1 we know

$$e_1 e_1 = e_1$$

$$\Rightarrow \qquad f(e_1 \cdot e_1) = f(e_1)$$

But $f(e_1 \cdot e_1) = f(e_1) f(e_1)$ by hypothesis (f respects multiplications) $\Rightarrow$

$$f(e_1) f(e_1) = f(e_1)$$

Multiply across by $f(e_1)^{-1}$ (which exists because G_2 is a group) to get

$$f(e_1) f(e_1) f(e_1)^{-1} = f(e_1) f(e_1)^{-1}$$

$$f(e_1) \cdot e_2 = e_2$$

$$\Rightarrow \qquad f(e_1) = e_2 \qquad \text{and (1) is proven.}$$

Proof of (2).

By definition of inverse,

$$x x^{-1} = x^{-1} x = e_1$$

$$\Rightarrow \quad f(x x^{-1}) = f(x^{-1} x) = f(e_1)$$

$$\Rightarrow \quad f(x) f(x^{-1}) = f(x^{-1}) f(x) = e_2$$

by hypothesis on f (respects multiplication) and property (1).

Multiply across by $f(x)^{-1}$ (which exists because G_2 is a group) and we get

$$f(x^{-1}) . f(x) . f(x)^{-1} = e_2 . f(x)^{-1}$$

$$\Rightarrow \quad f(x^{-1}) . e_2 = f(x)^{-1}$$

$$\Rightarrow \quad f(x^{-1}) = f(x)^{-1}$$ and (2) is proven.

$\boxed{//}$

Rk so functions between groups which respect multiplication must send identity to identity, and send inverses to inverses.
