Qn Find an isomorphism from $(\mathbb{Z}_5 - \{0\}, \times)$ to $(\mathbb{Z}_4, +)$

Answer $\mathbb{Z}_5 - \{0\}$ has 4 elements : $1, 2, 3, 4$

and $\mathbb{Z}_4$ has 4 elements : $0, 1, 2, 3$.

~~Z~~ We want to find a bijection $f : \mathbb{Z}_5 - \{0\} \longrightarrow \mathbb{Z}_4$ which respects the operations of multiplication and addition.

$$f(x\,y) = f(x) + f(y) \quad \longleftarrow (*)$$

Now 1 is the identity element in $\mathbb{Z}_5 - \{0\}$.

$$1.1 = 1$$

$$\Rightarrow \quad f(1.1) = f(1)$$

By property $(*)$ this rewrites as

$$f(1) + f(1) = f(1)$$

Subtracting $f(1)$ (remember we are working with <u>addition</u> for the outputs) gives

$$f(1) + \cancel{f(1) - f(1)} = f(1) - f(1) = 0$$

$$\Rightarrow \quad \boxed{f(1) = 0.} \quad \longleftarrow f \text{ must take } 1 \text{ to } 0.$$

Look at "squares of elements"

$1.1 = 1$ $\quad$ we already know $f(1) = 0$ $\quad$ $0+0 = 0$

$2.2 = 4$ $\qquad$ $1+1 = 2$

$3.3 = 4$ $\qquad$ $2+2 = 0$

$4.4 = 1$ $\qquad$ $3+3 = 2$

$(\mathbb{Z}_5 - \{0\}, \times)$

$f(4.4) = f(1)$ rewrites (using (∗)) as

$$f(4) + f(4) = f(1) = 0$$

$$f(4) + f(4) = 0.$$

Looking at the right hand column, we see that the only possibilities are $f(4) = 0$ $\quad (0+0=0)$

and $f(4) = 2$ $\quad (2+2=0)$

But $f(1) = 0$ & f is injective.

$$\Rightarrow f(4) \neq 0$$

$$\Rightarrow \boxed{f(4) = 2}$$ f <u>must</u> send 4 to 2.

There are only 2 possibilities left:

$f(2) = 1$, $f(3) = 3$ // or $f(2) = 3$, $f(3) = 1$

Both work!

$$f_1 : (\mathbb{Z}_5 - \{0\}, \times) \longrightarrow (\mathbb{Z}_4, +)$$

$$\begin{aligned} : 1 &\longmapsto 0 \\ 2 &\longmapsto 1 \\ 3 &\longmapsto 3 \\ 4 &\longmapsto 2 \end{aligned}$$

$$\& \quad f_2 : (\mathbb{Z}_5 - \{0\}, \times) \longrightarrow (\mathbb{Z}_4, +)$$

$$\begin{aligned} : 1 &\longmapsto 0 \\ 2 &\longmapsto 3 \\ 3 &\longmapsto 1 \\ 4 &\longmapsto 2 \end{aligned}$$

Remark It is perhaps more natural to write out the inverse functions

$$f_1^{-1} : (\mathbb{Z}_4, +) \longrightarrow (\mathbb{Z}_5 - \{0\}, \times)$$

$$: x \longmapsto 2^x \qquad \left(\text{exponential function!}\right)$$

$$f_2^{-1} : (\mathbb{Z}_4, +) \longrightarrow (\mathbb{Z}_5 - \{0\}, \times)$$

$$: x \longmapsto 3^x \qquad \left(\text{exponential function!}\right)$$