Explicit verification of Lagrange's Theorem proof for some subgroups of $\mathrm{Perm}(\{1,2,3\})$.

$$G = \mathrm{Perm}(\{1,2,3\}) = \{1, (12), (13), (23), (123), (132)\}$$

eg 1 $\quad H_1 = \{1, (12)\} < G$.

$$\begin{array}{lcl} & & G \\ 1H_1 = (12)H_1 & \longrightarrow & \boxed{1 \quad (12)} \\ (23)H_1 = (132)H_1 & \longrightarrow & \boxed{(23) \quad (132)} \\ (13)H_1 = (123)H_1 & \longrightarrow & \boxed{(13) \quad (123)} \end{array}$$

- $|L_g(H_1)| = |H_1|$ for each $g$ (since $L_g$ injective)
- The $L_g(H_1)$ sets are either equal or disjoint (empty intersection).

$\Rightarrow$ they partition $G$ into 3 disjoint sets each with $|H_1|$ elements.

$$\Rightarrow 3|H_1| = |G|$$

$$\Rightarrow |H_1| \,\big|\, |G|$$

eg ② $H_2 = \{1, (123), (132)\} < G.$

$G$

| | | | |
|---|---|---|---|
| $H_2 = (123)H_2 = (132)H_2 \longrightarrow$ | $1$ | $(123)$ | $(132)$ |
| $(12)H_2 = (13)H_2 = (23)H_2 \longrightarrow$ | $(12)$ | $(23)$ | $(13)$ |

- $|L_g(H_2)| = |H_2|$ for each $g$ (since $L_g$ injective).
- The $L_g(H_2)$ sets are either equal or disjoint

$\Rightarrow$ they partition $G$ into 2 disjoint sets, each with $|H_2|$ elements.

$$\Rightarrow \quad 2|H_2| = |G|$$

$$\Rightarrow \quad |H_2| \,\Big|\, |G|$$

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**Remark** $H_1$ = subgroup, $\langle (12) \rangle$, generated by all powers of $(12)$.

$\Rightarrow$ order$((12)) = |H_1|$, and so order$(12) \,|\, |G|$.

$H_2$ = subgroup, $\langle (123) \rangle$, generated by all powers of $(123)$.

$\Rightarrow$ order$((123)) = |H_2|$, and so order$((123)) \,|\, |G|$.