e is not a rational number.

We start with the Maclaurin series for e^x . . . and set $x=1$.

$$e = e^1 = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e = \left(1 + \cdots + \frac{1}{n!}\right) + \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right)$$

Subtract this from both sides

$$e - \left(1 + \cdots + \frac{1}{n!}\right) = \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right)$$

This is positive. Hence we can add a "0<" prefix . . . .

$$0 < e - \left(1 + \cdots + \frac{1}{n!}\right) = \left(\frac{1}{(n+1)!} + \cdots\right)$$

$$= \frac{1}{n!}\left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\right)$$

because $\frac{1}{n+2} < \frac{1}{n+1}$, $\frac{1}{n+3} < \frac{1}{n+1}$, $\vdots$ $\longrightarrow$

$$< \frac{1}{n!}\left(\frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots\right)$$

Convergent Geom. Series with $a = r = \frac{1}{n+1}$

$$\Rightarrow \text{Sum} = \frac{a}{1-r} = \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{\cancel{n+1}} \cdot \frac{\cancel{n+1}}{n} = \frac{1}{n}$$

Thus

$$0 < e - \left(1 + \cdots + \frac{1}{n!}\right) < \frac{1}{n!} \cdot \frac{1}{n}$$

Multiply across by the positive number $n!$ to get

$$\boxed{0 < n!\,e - n!\left(1 + \cdots \frac{1}{n!}\right) < \frac{1}{n}} \quad \text{——}(*)$$

Assume that the number e is rational. That is

$$e = p/q \quad \text{for some positive integers } p, q.$$

Now choose an integer $n \geq q$.

$$n!\,e = n!\,\frac{p}{q}$$ is an integer since q is a factor of $n!$

Also $$n!\left(1 + \cdots + \frac{1}{n!}\right) = n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!}$$

is an integer.

Thus $n!\,e - n!\left(1 + \cdots + \frac{1}{n!}\right)$ is an integer. But $(*)$ implies that this integer lies strictly between 0 & $\frac{1}{n} < 1$. This is impossible!

Thus the assumption that $e = p/q$ for some positive integers p, q is wrong.

Hence e is irrational.