PRINT NAME: <u>SOLUTIONS</u>

Calculus IV [2443–002] Midterm III

1 Q1 [15 points]

1.1 Part 1

Write down the change of variables formula for triple integrals.

1.2 Answer to part 1

Suppose the change of variables $(x(u,v,w), y(u,v,w), z(u,v,w))$ takes a region S in uvw-space to a region R in xyz-space. Then

$$\iiint_R f(x,y,z)\,dV \;=\; \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \;=\; \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

1.3 Part 2

Use the change of variables formula to evaluate the volume of the ellipsoid bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \;=\; 1$$

Show all the steps of your work clearly. You may use the fact that the volume of a sphere of radius r is equal to $\frac{4}{3}\pi r^3$.

1.4 Answer to part 2

Recall that

$$\text{Volume} \;=\; \iiint_{\text{ellipsoid}} dV$$

Let $x = au$, $y = bv$, and $z = cw$. Then the ellipsoid above becomes a unit ball bounded by the sphere $u^2 + v^2 + w^2 = 1$ in uvw-space. We also have $x_u = a$, $x_v = x_w = 0$, $y_v = a$, $y_u = y_w = 0$, and $z_w = a$, $z_u = z_v = 0$. Thus we get

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \;=\; \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \;=\; abc$$

and, substituting into the change of variables formula yields

$$\iiint_{\text{ellipsoid}} dV \;=\; \iiint_{\text{unit ball}} abc\,du\,dv\,dw \;=\; abc(\text{Volume of unit ball}) \;=\; \frac{4\pi}{3}abc\,.$$

2 Q2 [15 points]

2.1 Part 1

Write down the expression for volume element dV in spherical coordinates. [Recall $dV = dxdydz$ in Cartesian coordinates]

2.2 Answer to part 1

The volume element in spherical coordinates is just

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

2.3 Part 2

Use spherical coordinates to compute the volume of the solid which lies below the sphere $x^2 + y^2 + z^2 = 9$ and above the (upper half ($z > 0$) of the) cone $z^2 = 3(x^2 + y^2)$.

2.4 Answer to part 2

The equation of the cone is $z^2 = 3r^2$ or $z = \sqrt{3}r$ since we're looking at upper half of cone. The cone angle (between positive z-axis and slant surface of cone) is equal to the upper limit of ϕ and is given by $\tan\phi = r/z = 1/\sqrt{3}$. Thus ϕ has a upper limit of $\pi/6$, and the solid has the following spherical coordinates description

$$0 \le \rho \le 3\,; \qquad 0 \le \theta \le 2\pi\,; \qquad 0 \le \phi \le \pi/6$$

Therefore the volume is given by

$$\int_0^{\pi/6}\int_0^{2\pi}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \left[\rho^3/3\right]_0^3 [\theta]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/6} = (9)(2\pi)(1-\sqrt{3}/2) = 18\pi(1-\sqrt{3}/2)\,.$$

3 Q3 [15 points]

All parts of this question refer to the vector field

$$\mathbf{F} = \langle 2zx + \sin y, x\cos y, x^2 \rangle\,.$$

3.1 Part 1

Compute $curl\mathbf{F}$.

3.2 Answer to part 1

We recall that

$$curl\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2zx + \sin y & x\cos y & x^2 \end{vmatrix} = \langle\, 0-0,\, 2x-2x,\, \cos y - \cos y\,\rangle = \langle\, 0,\, 0,\, 0\,\rangle$$

3.3 Part 2

Does there exist a function $\phi(x, y, z)$ such that $\nabla\phi = \mathbf{F}$? If so, find such a function ϕ.

3.4 Answer to part 2

Since $curl\mathbf{F} = \mathbf{0}$ and $\mathbf{F}$ is defined on all of $\mathbf{R}^3$ we conclude that $\mathbf{F}$ is conservative. That is, there does indeed exist a function $\phi(x, y, z)$ satisfying $\nabla\phi = \mathbf{F}$.

We want $\langle \phi_x, \phi_y, \phi_z \rangle = \mathbf{F}$. Thus we must solve the following equations

$$\phi_x = 2zx + \sin y\,; \qquad \phi_y = x\cos y\,; \qquad \phi_z = x^2\,.$$

The first equation gives $\phi = x^2z + x\sin y$ plus some function of y and z. It is easy to see from the second and third equations that this function is a constant, and so we get

$$\phi(x, y, z) = x^2z + x\sin y + C\,.$$

3.5 Part 3

Determine the value of the path integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the path given by $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$ for $0 \le t \le 3\pi/4$.

3.6 Answer to part 3

By the fundamental Theorem of Path Integrals we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla\phi \cdot d\mathbf{r} &= \phi(\mathbf{r}(3\pi/4)) - \phi(\mathbf{r}(0)) \\ &= \phi(\sqrt{2}/2, 3\pi/4, -\sqrt{2}/2) - \phi(0, 0, 1) \\ &= -\sqrt{2}/4 + 1/2\,. \end{aligned}$$

4 Q4 [15 points]

4.1 Part 1

Write down the equation that appears in Green's theorem, stating what each part means.

4.2 Answer to part 1

Green's Theorem states that

$$\int_C P\,dx + Q\,dy = \iint_R [Q_x - P_y]\,dA$$

where C is a smooth, positively oriented, simple, closed curve which bounds a region R in the plane, and P and Q have continuous partial derivatives on R.

4.3 Part 2

Use Green's Theorem to show that if C is a smooth, positively oriented, simple, closed curve then $\int_C xdy$ represents the area of the region enclosed by C.

4.4 Answer to part 2

It's basically an immediate application of Green.

$$\int_C x\,dy \;=\; \int_C 0\,dx \,+\, x\,dy \;=\; \iint_R \left[\frac{\partial x}{\partial x} \,-\, \frac{\partial 0}{\partial y}\right] dA \;=\; \iint_R 1\,dA$$

which is precisely the area of the region R enclosed by C.

4.5 Part 3

Use Green's theorem to evaluate the path integral of the vector field $\mathbf{F} = \langle xy, y^5 \rangle$ around the positively oriented triangle C with vertices $(0,0)$, $(2,0)$, and $(2,1)$.

4.6 Answer to part 3

Let T denote the triangular region enclosed by C. Then Green's Theorem states that

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_T \left[\frac{\partial(y^5)}{\partial x} - \frac{\partial(xy)}{\partial y}\right] dA \\
&= \iint_T -x\,dA \\
&= -\int_0^1 \int_{2y}^2 x\,dx\,dy \\
&= -\int_0^1 \left[x^2/2\right]_{2y}^2 dy \\
&= \int_0^1 [2y^2 - 2]\,dy \\
&= \left[2y^3/3 - 2y\right]_0^1 \\
&= -4/3
\end{aligned}$$

5 Bonus

You have just been hired by the POCKETGOPHER mining and tunneling company to help them design their next generation drill bit. The basic drill design consists of a cylindrical axis with a helical blade as shown in the diagram. Now the people at POCKETGOPHER know that the friction on the drill bit depends on surface area, and they know the surface area of a cylinder, but they need your calculus expertise when it comes to helical surfaces.

5.1 Part 1

Give a parametric description of the helical surface with helix boundary curves given by $\langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 2\pi$ and by $\langle 2\cos t, 2\sin t, t \rangle$ for $0 \leq t \leq 2\pi$. You should think of this surface as being made up of *horizontal* (parallel to xy-plane) line segments which begin on one helix and end on the other. Something like a spiral staircase with infinitely many (infinitesimally small) steps!

5.2 Answer to part 1

The hint tells us to consider the surface as being made up of infinitely many horizontal line segments with one endpoint on the first helix and the other endpoint on the second helix. So the horizontal line segment at height t begins at the point $(\cos t, \sin t, t)$ and ends at the point $(2\cos t, 2\sin t, t)$. Thus points on this line segment may be parameterized as follows

$$\langle 0, 0, t\rangle \;+\; s\langle \cos t, \sin t, 0\rangle \qquad\qquad \text{for } 1 \le s \le 2.$$

This gives the following parametric description for the helical surface

$$\langle x(s,t), y(s,t), z(s,t)\rangle \;=\; \langle s\cos t, s\sin t, t\rangle \qquad\qquad \text{for } 1 \le s \le 2 \text{ and } 0 \le t \le 2\pi.$$

5.3 Part 2

Compute the surface area of the helical surface that you parameterized above.

5.4 Answer to part 2

Recall that surface area is given by the double integral

$$\iint \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(s,t)}\right)^2}\, ds\, dt$$

over the rectangular region $[1,2]\times[0,2\pi]$ in the st-plane. So we have to compute the three determinants first of all.

Now,

$$\frac{\partial(x,y)}{\partial(s,t)} \;=\; \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} \;=\; \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} \;=\; s$$

and

$$\frac{\partial(y,z)}{\partial(s,t)} \;=\; \begin{vmatrix} y_s & y_t \\ z_s & z_t \end{vmatrix} \;=\; \begin{vmatrix} \sin t & s\cos t \\ 0 & 1 \end{vmatrix} \;=\; \sin t$$

and

$$\frac{\partial(z,x)}{\partial(s,t)} \;=\; \begin{vmatrix} z_s & z_t \\ x_s & x_t \end{vmatrix} \;=\; \begin{vmatrix} 0 & 1 \\ \cos t & -s\sin t \end{vmatrix} \;=\; -\cos t\,.$$

This gives a surface area integral of

$$\int_0^{2\pi}\int_1^2 \sqrt{s^2+1}\, ds\, dt \;=\; 2\pi\int_1^2 \sqrt{s^2+1}\, ds\,.$$

You would get full marks for this much. You may evaluate this last integral by a trig substitution (or by looking it up in the table) to get a final answer of $2\pi[s/2\sqrt{s^2+1} + 1/2\ln(s+\sqrt{s^2+1})]_1^2$.