

A vector field (in 2 or 3-dimensions) is just defined as

$$\mathbf{F} = \langle P(x, y), Q(x, y) \rangle \quad \text{in 2-dimensions,}$$

and

$$\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \quad \text{in 3-dimensions.}$$

Vector fields in real life: We encounter many examples of vector fields in real life. Let's think about what people might be interested in vector fields and why. We can (without too much thought) come up with two different interpretations of the abstract mathematical notion of a vector field.

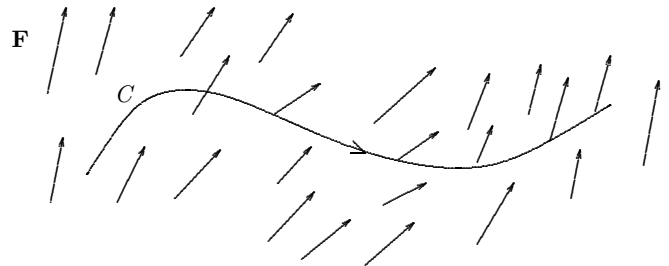
1. **Force fields.** In your physics class you hear about *force fields*. These can be electrostatic (associated to a charged particle), magnetic, gravitational etc.
2. **Velocity vector fields.** Engineers and meteorologists are often interested in *velocity vector fields*. These fields describe *fluid flow, wind currents, etc.*

Integration of vector fields. The various physical interpretations of vector fields lead us to consider different ways of *integrating* vector fields.

1. **Path integrals (Line integrals).** In physics we are interested in the *work done* by a force field \mathbf{F} as it moves a particle along a path C . To compute this, we partition the path C , and then sum up the *tangential components* of the force field times the arc length elements. On taking finer and finer partitions these (Riemann) sums have a limit which we interpret as the work done by \mathbf{F} along the path C .

WORK DONE BY \mathbf{F}

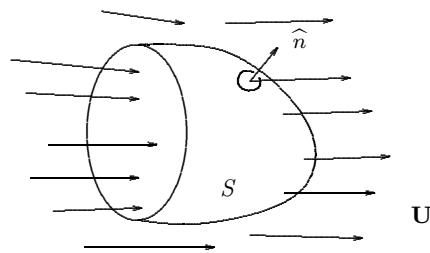
$$= \int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds$$



2. **Surface integrals.** A basic quantity in fluid mechanics is the net volume of fluid which flows through a surface S in unit time. This is also called the *flux* of the velocity field of the fluid through S . We compute this by first partitioning the surface into small area elements. The net volume of fluid flowing through a given area element in unit time is just given by the component of the fluid velocity \mathbf{U} which is perpendicular to (normal to) this area element times the area of the element. Again we sum and then take a limit of the resulting Riemann sums to get

FLUX OF \mathbf{U} THROUGH S

$$= \iint_S \mathbf{U} \cdot \hat{\mathbf{n}} dA$$



There are three basic steps in computing these integrals.

1. Give a description of the path C and its tangent vectors $\hat{\mathbf{T}}$ in the case of path integrals, and give a description of the surface S and its normal vectors $\hat{\mathbf{n}}$ in the case of surface integrals. We already know how to describe curves in 2 and 3-dimensions parametrically, and how to compute their unit tangent vectors $\hat{\mathbf{T}}$ and arclength elements ds [remember Calc III]. Later on, we will develop an analogous parametric description for surfaces in 3-d, and we'll compute their unit normal vectors $\hat{\mathbf{n}}$ and area elements dA .
2. Use the definitions on the previous stage together with the descriptions above to write down explicit 1 or 2-variable integrals. This is a purely formal step and does not require any work.
3. Compute the resulting definite integrals (or double integrals). This can take a bit of work [mainly Calc II].

Basically, that's all there is to integrating vector fields in 2 and 3-dimensions! If you keep the physical interpretations of force fields and velocity vector fields in mind, the definitions of line integral and surface integral will seem very natural. Finally, remember that, although there is a bit of work involved in computing these integrals, most of the computational work has been developed in the Calc II and Calc III courses.