MATH 2443-004 EXTRA HOMEWORK II FALL 2003

Weird and wonderful applications of double integrals.

Problem 1. Call a rectangle in the euclidean plane *good* if at least one of its length or width is a rational number. For example, the $2 \times \pi$ and $3/4 \times 7/9$ rectangles are both good, while a $\sqrt{2} \times e$ rectangle is not good. Suppose you are given a big rectangle R, and you are told that it is somehow tiled by a finite number of good rectangles. Prove that R is also good.

Note that this tiling can have a small $2 \times \pi$ rectangle (good in the horizontal direction) adjacent to a $\sqrt{2} \times 4/5$ rectangle (which is good in the vertical direction). You have to somehow conclude that, since everything fits together in a nice finite tiling, the sum of horizontal lengths

$$\cdots + 2 + \sqrt{2} + \cdots$$

is rational, or that the sum of vertical lengths

$$\cdots + \pi + \cdots$$

is rational (or possibly that both are rational).

Problem 2. Prove the following result that we saw in Calc III.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Ideas for problem 1.

(1) Call a rectangle *really good* if one of its lengths is an integer. Given our big rectangle R tiled by finitely many small good rectangles, we can take L to be the least common multiple of the denominators of all the rational lengths of the small good rectangles. Scale everything by L, so now we get a huge rectangle, H, which is tiled by really good rectangles. We'll argue that this huge rectangle is also really good (and hence we can conclude that R is good — why?).

(2) Consider the function $f(x,y) = \sin(\frac{x}{2\pi})\sin(\frac{y}{2\pi})$. What is the value of

$$\iint f(x,y)\,dA$$

over a really good rectangle? Why?

(3) What is value of the double integral of f over H? Why?

(4) Place H so that its bottom left corner is at the origin, so it has two other corners at $(A,0)$ and $(0,B)$ say. What can you conclude about A or B? Why?

(5) Why did we have to be careful about the placement of the bottom left corner of H? What other points would work? What would be a bad placement for H?

Ideas for problem 2. This is basically an evaluation of the double integral of $f(x,y) = \frac{1}{1-xy}$ over the unit square $[0,1]\times[0,1]$ in two different ways.

(1) First method. Use the geometric series

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$$

to write out the integrand above as a series in x and y, and then integrate this series term by term, first with respect to x and then with respect to y. You can just do this formally if you like. However, if you have issues with the fact that the integral is not proper ($f(1,1)$ is not defined), and/or that a geometric series only makes sense for $|r| < 1$, then you should work all this for the rectangle $[0,1]\times[0,a]$ for $a < 1$, and then consider what happens as $a \to 1$.

(2) Second Method. (Rotate coordinates by $\pi/4$!) Use the substitution

$$x = \frac{u-v}{\sqrt{2}}\,, \qquad y = \frac{u+v}{\sqrt{2}}$$

(3) Show that $xy = \frac{(u^2-v^2)}{2}$, and that $1 - xy = \frac{(2-u^2+v^2)}{2}$

(4) Show that the integral becomes the sum of two integrals

$$4\int_0^{\sqrt{2}/2}\int_0^u \frac{1}{2-u^2+v^2}\,dv\,du \;+\; 4\int_{\sqrt{2}/2}^{\sqrt{2}}\int_0^{\sqrt{2}-u} \frac{1}{2-u^2+v^2}\,dv\,du$$

(5) The v-integrals will give you inverse tangents. To do the resulting horrible u-integrals, make the substitution $u = \sqrt{2}\sin\theta$, so that $\sqrt{2-u^2} = \sqrt{2}\cos\theta$ and $du = \sqrt{2}\cos\theta d\theta$. Have fun.