MATH 4513-001 | Senior Seminar | Spring 2010

Homework Set 3

Please complete by class time on Thursday, Feb 25.

1. Write down five different elements $g \in S_5$ which conjugate $(12)(34)$ into $(13)(24)$. That is, find five different elements $g \in S_5$ which satisfy the equation

$$g(12)(34)g^{-1} = (13)(24)$$

2. Write down a detailed argument to show that the $(n-1)n/2$ transpositions (pq) for $1 \leq p < q \leq n$ generate all of S_n.

3. Write down a detailed argument to show that the two elements (12) and $(1 \ldots n)$ generate all of S_n.

4. Verify that $\{(12), (123)\}$, $\{(12), (23)\}$ are two generating sets for S_3. Also, draw the Cayley graphs of S_3 with respect to these two generating sets. You should draw two separate graphs.

5. Compare the Cayley graph of S_3 with respect to $\{(12), (123)\}$ with the Cayley graph of $\mathbb{Z}_6$ with respect to $\{2, 3\}$. Any similarities? Any differences?

6. Draw the Cayley graph of S_4 with respect to the generating set $\{(12), (23), (34)\}$ (also verify that this is indeed a generating set).

BASIC FACT ABOUT CONJUGATION....

1. $$g\,(12)(34)\,g^{-1} = (g(1)\ g(2))\,(g(3)\ g(4))$$

Therefore $g\,(12)(34)\,g^{-1} = (13)(24)$

becomes

$$(g(1)\ g(2))\,(g(3)\ g(4)) = (13)\,(24)$$

$g(1)=1$, $g(3)=2$ $g(2)=3$, $g(4)=4$ eg $g=(23)(1)(4)(5)$	$g(1)=3$, $g(3)=2$ $g(2)=1$, $g(4)=4$ eg $g=(132)(4)(5)$
$g(1)=1$, $g(3)=4$ $g(2)=3$, $g(4)=2$ eg $g=(234)(1)(5)$	$g(1)=3$, $g(3)=4$ $g(2)=1$, $g(4)=2$ $g=(2134)(5)$

There are <u>NO</u> other choices for g! In all 4 cases, the effect of g on the remaining elements (including 5) ~~are~~ is completely determined.

There are only 4 such g.

2. Effect of a permutation on $\{1, \dots, n\}$ is to permute elements around in disjoint cycles. ...
In cycle notation

$$\text{perm} = (m_1 \dots m_k)(p_1 \dots p_l) \dots$$

So it suffices to prove that each cycle can be written as a product of transpositions.

But $(a_1 \dots a_k) = (a_1 a_2)(a_2 a_3) \dots (a_{k-1} a_k)$.

3. By Q2, it is sufficient to show that all $n(n-1)/2$ transpositions can be obtained from (12) and $g = (12 \dots n)$.

$$\left.\begin{aligned} g(12)g^{-1} &= (23) \\ g^2(12)g^{-2} &= (34) \\ &\vdots \\ g^{n-2}(12)g^{-(n-2)} &= (n-1, n) \\ g^{n-1}(12)g^{-(n-1)} &= (n\ 1) \end{aligned}\right\} \Rightarrow \text{have } (12), (23), \dots, (n-1, n), (n\ 1)$$

$$\begin{aligned} (12)(23) &= (123) &&= h_2 \\ (12)(23)(34) &= (1234) &&= h_3 \\ &\vdots \\ (12)(23) \dots (n-2\ n-1) &= (12 \dots n-1) &&= h_{n-2} \end{aligned}$$

Note that $h_2^{-1}(12)h_2 = (31) = (13)$
$h_3^{-1}(12)h_3 = (4\,1) = (14)$
$\vdots$
$h_{n-2}^{-1}(12)h_{n-2} = (n-1\;1) = (1\;n-1)$ — [+]

add in (12) & (1n) from earlier.

Now, to get $(p\,q)$ $p<q$ we simply conjugate

$(1\;(q-p+1))$ from list [+]

by g^{p-1}

$$g^{p-1}\,(1\;(q-p+1))\,g^{-(p-1)} = (p\,q)$$

Recall $g = (12\ldots n)$

So now we have obtained all $n(n-1)/2$ transpositions $(p\,q)$ $p<q$ from (12) & $(12\ldots n)$.

Since the transpositions generate S_n $\Rightarrow$ so does $\{(12),(123\ldots n)\}$

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4. $\{(12), (123)\}$ works by Q3 ($n=3$ case).

Given $\{(12), (23)\}$ $\Rightarrow$ can get $(123) = (12)(23)$

$\Rightarrow$ $\{(12), (123)\}$ generates same subgroup as $\{(12), (123)\}$

i.e. all of S_3.

Cayley graphs drawn in class!

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5. Done in class!

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6. Done in class!

Note: $(12)(23)(34) = (1234)$ $\Rightarrow$ we can get $(12), (1234)$ & hence all of S_4 from $(12), (23), (34)$.
