SP '11 - MATH 3333 - MID I - SOLUTIONS

Q1]... [20 points] Say whether each of the following statements is True or False.

(1) The equation $(AB)^T = B^T A^T$ holds for all $n \times n$ matrices A and B.

TRUE

(2) The matrix transformation given by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates vectors in $\mathbb{R}^2$ clockwise through θ radians about the origin.

FALSE (it's a <u>counterclockwise</u> rotation)

(3) If an $n \times n$ matrix A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

TRUE

(4) The equation $AB = BA$ holds for all $n \times n$ matrices A and B.

FALSE eg: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(5) If the $n \times n$ matrix A is singular, then every linear system $A\mathbf{x} = \mathbf{b}$ with coefficient matrix A has infinitely many solutions.

FALSE

Not every. Some may have <u>NO SOLUTION</u>

eg $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

Row of ZEROS

NON ZERO RHS

Q2]... **[20 points]** Write down the augmented matrix for the following system.

$$\begin{aligned} y - 8z &= -17 \\ x + z &= 10 \\ x - y &= 0 \end{aligned}$$

$$\begin{aligned} 0.x + 1.y + (-8)z &= -17 \\ 1.x + 0.y + 1.z &= 10 \\ 1.x + (-1)y + 0.z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -8 & -17 \\ 1 & 0 & 1 & 10 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

Now, solve the system using the method of **Gaussian elimination**.

$$\underset{r_1 \leftrightarrow r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 0 & 1 & -8 & -17 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$\underset{r_3 + (-1)r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 0 & 1 & -8 & -17 \\ 0 & -1 & -1 & -10 \end{array}\right]$$

$$\underset{r_3 + r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & -9 & -27 \end{array}\right]$$

$$\underset{-\frac{1}{9}(r_3)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

In Row Echelon form

$z = 3$

$x + z = 10 \Rightarrow x = 10 - 3$

$\Rightarrow x = 7$

$y - 8z = -17$

$y - 8(3) = -17$

$y = 24 - 17 = 7$ $\quad y = 7$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 3 \end{pmatrix}}$$

Q3]... **[20 points]** Using the row reduction algorithm discussed in class, determine if the following matrix A is invertible. If it is, find its inverse.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{r_1 \leftrightarrow r_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{r_2 - r_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{r_2 \leftrightarrow r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\underset{r_3 - 3(r_2)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -1 & -3 \end{array}\right]$$

$$\underset{\frac{r_3}{-2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array}\right]$$

$$\underset{r_2 - r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array}\right]$$

I_3 $\Rightarrow$ A is invertible and $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Q4]. . . **[20 points]** Write the following matrix A as a product of elementary matrices. Show your work

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\underset{r_2 - 2(r_1)}{\sim} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \quad \text{---} \quad E_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\underset{-1(r_2)}{\sim} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad \text{---} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\underset{r_1 - r_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad \text{---} \quad E_3 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

So A^{-1} exists & is given by

$$A^{-1} = E_3 E_2 E_1 I_2 = E_3 E_2 E_1$$

$$\Rightarrow \quad A = (E_3 E_2 E_1)^{-1}$$

Note: Reverse ORDER

$$= E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ +2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & +1 \\ 0 & 1 \end{pmatrix}$$

= Product of elementary matrices.

Remark: There are many solutions (depending on how you row reduce A to I_2) but it'll take at least 3 row ops $\Rightarrow$ product will be at least 3 elem. matrices.

eg $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ eg $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

eg $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ also work!

Q5]...[20 points] Let A and B be 2×2 matrices, $\mathbf{u}$ and $\mathbf{v}$ be 2×1 column vectors, and $\mathbf{0}$ be the 1×2 row vector. Prove that the following 3×3 matrices multiply as shown.

$$\begin{pmatrix} A & \mathbf{u} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} B & \mathbf{v} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} AB & A\mathbf{v}+\mathbf{u} \\ \mathbf{0} & 1 \end{pmatrix}$$

$$LHS = \left[\begin{array}{cc|c} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} b_{11} & b_{12} & v_1 \\ b_{21} & b_{22} & v_2 \\ \hline 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+0 & a_{11}b_{12}+a_{12}b_{22}+0 & a_{11}v_1+a_{12}v_2+u_1 \\ a_{21}b_{11}+a_{12}b_{21}+0 & a_{21}b_{12}+a_{22}b_{22}+0 & a_{21}v_1+a_{22}v_2+u_2 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \hline 0 \quad 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} AB & A\vec{v}+\vec{u} \\ \hline 0 & 1 \end{array}\right] = RHS.$$

If A is invertible, then the 3×3 matrix $\begin{pmatrix} A & \mathbf{u} \\ \mathbf{0} & 1 \end{pmatrix}$ is also invertible. Write down its inverse

By Mult$^{\underline{n}}$ formula above, we Guess

A^{-1} goes in here!

$$\left[\begin{array}{c|c} A^{-1} & \vec{v} \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} A & \vec{u} \\ \hline 0 & 1 \end{array}\right]$$

want

$$= \left[\begin{array}{c|c} A^{-1}A & A^{-1}\vec{u}+\vec{v} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} I_2 & \vec{0} \\ \hline 0 & 1 \end{array}\right]$$

$\Rightarrow$ want $A^{-1}\vec{u}+\vec{v} = 0 \Rightarrow \vec{v} = -A^{-1}\vec{u}$

Ans $\left[\begin{array}{c|c} A & \vec{u} \\ \hline 0 & 1 \end{array}\right]^{-1} = \left[\begin{array}{c|c} A^{-1} & -A^{-1}\vec{u} \\ \hline 0 & 1 \end{array}\right]$