MATH 2443–008 Calculus IV Spring 2014

Notes on Stokes' Theorem

1. **Equation in Stokes' Theorem.** Read the textbook for the conditions on $\mathbf{F}$, S and ∂S.

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \tag{1}$$

2. **Stokes implies Green**. Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field on a domain D with boundary C satisfying the conditions of Green's Theorem.

   - Consider $\mathbf{F} = \langle P, Q, 0 \rangle$ as a vector field in 3–dimensions, where P and Q are thought of as functions of (x, y, z) which do not explicitly involve the variable z. A computation gives $\nabla \times \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle$.
   - The region D can be viewed as a parametric surface in 3–dimensions, by $\mathbf{r}(x, y) = \langle x, y, 0 \rangle$. We find that $d\mathbf{S} = \hat{\mathbf{k}} dxdy$, and so $(\nabla \times \mathbf{F}) \cdot d\mathbf{S} = [Q_x - P_y]dxdy$.
   - Thus, the left hand side of equation (1) becomes

   $$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \;=\; \oint_{\partial S} \langle P, Q, 0 \rangle \cdot d\mathbf{r} \;=\; \oint_C P dx + Q dy \;=\; \oint_C \mathbf{F} \cdot d\mathbf{r}$$

   and the right hand side becomes

   $$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \;=\; \iint_D [Q_x - P_y] dxdy$$

   Combining these two gives Green's Theorem.

3. **Geometric definition of Curl.** Let $\mathbf{F}$ be a smooth vector field. Let $\hat{\mathbf{u}}$ be a unit vector based at a point P in space, and let C_t denote the circle of radius t centered on P in the plane normal to $\hat{\mathbf{u}}$. The circle C_t is oriented in a right hand fashion with respect to $\hat{\mathbf{u}}$; this circle is the boundary of a disk D_t centered at P. Then

$$\hat{\mathbf{u}} \cdot (\nabla \times \mathbf{F})_{(P)} \;=\; \lim_{t \to 0} \frac{\oint_{C_t} \mathbf{F} \cdot d\mathbf{r}}{\pi t^2} \tag{2}$$

We see this by replacing the line integral in the numerator by a surface integral over D_t and noting that $d\mathbf{S} = \hat{\mathbf{u}} dS$.

$$\lim_{t \to 0} \frac{\iint_{D_t} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}}{\pi t^2} \;=\; \lim_{t \to 0} \frac{\iint_{D_t} \hat{\mathbf{u}} \cdot (\nabla \times \mathbf{F})\, dS}{\pi t^2} \;=\; \hat{\mathbf{u}} \cdot (\nabla \times \mathbf{F})_{(P)}$$

Now equation (2) means that the projection of $(\nabla \times \mathbf{F})_{(P)}$ in the $\hat{\mathbf{u}}$ direction is equal to the (limit of the) circulation of $\mathbf{F}$ about P in the plane perpendicular to $\hat{\mathbf{u}}$ per unit area.

Note that the *maximum circulation of* $\mathbf{F}$ *about the point* P occurs in the plane with normal vector $(\nabla \times \mathbf{F})_{(P)}$.

4. **If $\mathbf{F} = \nabla \times \mathbf{G}$ then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for every closed surface S.** Here is the idea. First, cut a disk out of the surface S by cutting along a suitable simple closed curve C. Call the disk S_1 and the remainder S_2. It the boundary of S_1 is C, then the boundary of S_2 will be $-C$ (the curve C with the opposite orientation).

   Stokes' Theorem applied to the vector field $\mathbf{G}$ and the surface S_1 gives that

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{S_1} (\nabla \times \mathbf{G}) \cdot d\mathbf{S} \;\overset{Stokes}{=}\; \oint_{\partial S_1} \mathbf{G} \cdot d\mathbf{r} \;=\; \oint_C \mathbf{G} \cdot d\mathbf{r}$$

   and, applied to the vector field $\mathbf{G}$ and the surface S_2 gives that

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{S_2} (\nabla \times \mathbf{G}) \cdot d\mathbf{S} \;\overset{Stokes}{=}\; \oint_{\partial S_2} \mathbf{G} \cdot d\mathbf{r} \;=\; \oint_{-C} \mathbf{G} \cdot d\mathbf{r} \;=\; -\oint_C \mathbf{G} \cdot d\mathbf{r}$$

   Adding, we get

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \;=\; 0$$

   Alternative Method. You could also think of this as being an immediate consequence of Stokes' Theorem as follows.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S} \;\overset{Stokes}{=}\; \oint_{\partial S} \mathbf{G} \cdot d\mathbf{r} \;=\; 0$$

   because the surface S is closed, and so has empty boundary $\partial S = \emptyset$. Therefore the last integral is over an empty curve and so is automatically equal to 0. In you don't like this reasoning, you can still use the argument given in the paragraph above.

5. **Remark.** The Divergence Theorem *almost* gives the result above. If $\mathbf{F} = \nabla \times \mathbf{G}$, then $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{G}) = 0$. Therefore, *if the closed surface S is the boundary of a region E where $\mathbf{F}$ and its divergence are defined*, then the Divergence Theorem gives

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iiint_E (\nabla \cdot \mathbf{F})\, dV \;=\; \iiint_E 0\, dV \;=\; 0$$

   But it may happen that $\mathbf{F}$ and its divergence is not defined over all points of E. In this case, the Divergence Theorem will not apply. However, the argument in item 4 still works in this case.

   For example, the vector field

$$\mathbf{F} \;=\; \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} \qquad\qquad (x, y, z) \neq (0, 0, 0)$$

   safisfies

   - $\mathbf{F}$ and $\nabla \cdot \mathbf{F}$ are not defined at $(0, 0, 0)$.
   - $\nabla \cdot \mathbf{F} = 0$
   - Let S denote the unit sphere $x^2 + y^2 + z^2 = 1$. Then $d\mathbf{S} = \langle x, y, z\rangle dS$ and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_S \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} \cdot \langle x, y, z\rangle dS \;=\; \iint_S \frac{1^2}{1^3} dS \;=\; 4\pi$$

   - Therefore, by item 4 above, $\mathbf{F}$ is not the curl of another vector field.