Sp 2014 ; Mid II ; SOLUTIONS

Q1]... [20 points] Use the method of **Lagrange multipliers** to find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the constraint $x^2 + y^2 + z^2 = 12$.

$$g(x,y,z) = x^2+y^2+z^2$$

$$\boxed{\begin{array}{l} \nabla f = \lambda \nabla g \\ g = 12 \end{array}} \rightsquigarrow \boxed{\begin{array}{l} yz = \lambda 2x \\ xz = \lambda 2y \\ xy = \lambda 2z \\ x^2+y^2+z^2 = 12 \end{array}}$$

$$\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$$2y^2z = 2x^2z \qquad y^2 = z^2$$

$$y^2 = x^2$$

$$x^2 = y^2 = z^2 \Rightarrow 3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2$$

$$\Rightarrow \begin{cases} y = \pm 2 \\ z = \pm 2 \end{cases}$$

$$f(\pm 2, \pm 2, \pm 2) = (\pm 2)(\pm 2)(\pm 2)$$

$$= \begin{cases} 8 & \leftarrow \text{Max value} \\ \underline{\text{OR}} & \\ -8 & \leftarrow \text{Min value} \end{cases}$$

Q2]. . . [20 points] Evaluate the double integral

$$\int_0^1 \int_0^y (x+y)\,dx\,dy$$

$$\iint = \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^y dy$$

$$= \int_0^1 \left(\frac{y^2}{2} + y^2 - 0\right) dy$$

$$= \frac{3}{2}\int_0^1 y^2\, dy$$

$$= \frac{3}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{3} - 0\right) = \frac{3}{2}\left(\frac{1}{3}\right) = \boxed{\frac{1}{2}}$$

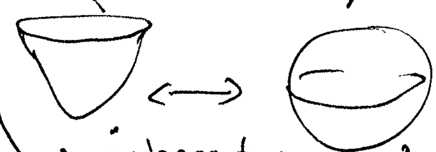
Q3]... [20 points] Use polar coordinates to write down a double integral expression for the volume of the region contained above the paraboloid $z = x^2 + y^2$ and below the sphere $x^2 + y^2 + z^2 = 12$. **You do not have to evaluate the double integral.**

below $z = \sqrt{12 - r^2}$ & above $z = r^2$

$\Rightarrow \sqrt{12 - r^2} - r^2$

$$Vol = \iint_{\substack{\text{Disk of} \\ \text{radius} \\ \sqrt{3}}} \left(\sqrt{12 - r^2} - r^2\right) dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \left(\sqrt{12 - r^2} - r^2\right) r \, dr \, d\theta$$



intersect in a circle

$z = x^2 + y^2 \Longrightarrow \boxed{z \geq 0}$

& $(x^2 + y^2) + z^2 = 12$

$\Downarrow$

$z + z^2 = 12$

$z^2 + z - 12 = 0$

$(z - 3)(z + 4) = 0$

$z = 3$ ..... since $z \geq 0$

$x^2 + y^2 = 3$

$\sqrt{x^2 + y^2} = \sqrt{3}$ Radius of circle

Write down a double integral in cartesian coordinates for the same volume above. **You do not have to evaluate the double integral.**

$$Vol = \iint_{\substack{\text{Disk of} \\ \text{radius } \sqrt{3}}} \left(\sqrt{12 - x^2 - y^2} - x^2 - y^2\right) dA$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3 - x^2}}^{\sqrt{3 - x^2}} \left(\sqrt{12 - x^2 - y^2} - x^2 - y^2\right) dy \, dx$$

Q4]... **[20 points]** The following triple integral describes the volume of a region in 3-dimensional space. **You do not have to evaluate the triple integral.** Describe the region (using pictures and inequalities).

$$\int_0^1 \int_y^1 \int_0^z dxdzdy$$

Inequalities

$0 \le x \le z$

$y \le z \le 1$

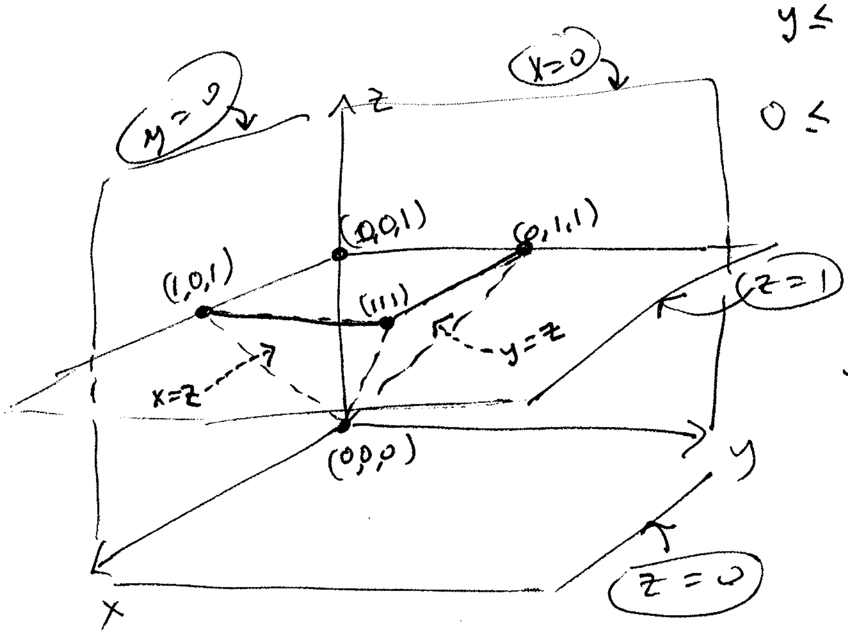
$0 \le y \le 1$

Planes

$x=0$; $x=z$

$z=y$; $z=1$

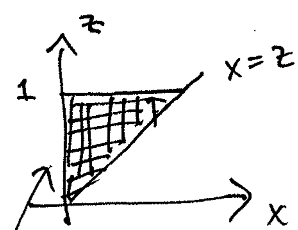
$y=0$; $y=1$



PYRAMID with vertices $(0,0,0)$, $(1,0,1)$, $(1,1,1)$, $(0,0,1)$ and $(0,1,1)$.

(with faces $z=1$; $x=0$; $y=0$; $x=z$; $y=z$)

Write down a triple integral for the volume above which changes the order of integration, using y first, then z, and finally x. **You do not have to evaluate the triple integral.**



$$\int_0^1 \int_x^1 dz\, dx$$

$0 \le y \le z$

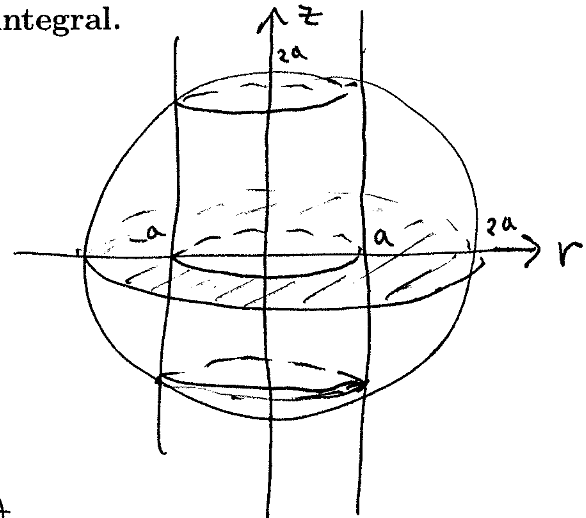
$y=0$ plane

$y=z$ plane

$\Downarrow$

$$\int_0^1 \left(\int_x^1 \left(\int_0^z dy \right) dz \right) dx$$

Q5]... [20 points] Use cylindrical coordinates to write down a triple integral for the volume of the solid region which lies inside of the sphere $x^2 + y^2 + z^2 = 4a^2$ but outside of the cylinder $x^2 + y^2 = a^2$. **You do not have to evaluate the triple integral.**



$$x^2 + y^2 + z^2 = 4a^2$$

$$z = \pm\sqrt{4a^2 - r^2}$$

$$\text{Vol} = \iint_{\text{Annulus}} \left(\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} 1 \, dz \right) dA$$

Annulus
(inner rad. = a; outer rad = 2a)

$$= \int_0^{2\pi} \left(\int_a^{2a} \left(\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} 1 \cdot dz \right) r \, dr \right) d\theta$$

Use spherical coordinates to write down a triple integral for the same volume above. **You do not have to evaluate the triple integral.**

$$\text{Vol} = \int_0^{2\pi} \left(\int_{\pi/6}^{5\pi/6} \int_{a\csc\phi}^{2a} \rho^2 \, d\rho \, \sin\phi \, d\phi \right) d\theta$$

Cylinder:

$$r^2 = a^2$$

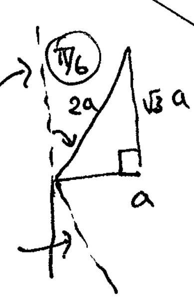
$$(\rho \sin\phi)^2 = a^2$$

$$\boxed{\rho = \frac{a}{\sin\phi} = a\csc\phi}$$

Sphere: $\boxed{\rho = 2a}$

θ angles = 0 through 2π

ϕ Angles



$\pi - \pi/6$

Miscellaneous Formulas.

- **Polar Coordinates.** $x = r\cos(\theta)$; $y = r\sin(\theta)$

$$dA = r dr d\theta$$

- **Cylindrical Coordinates.** $x = r\cos(\theta)$; $y = r\sin(\theta)$; $z = z$

$$dV = r dr d\theta dz$$

- **Spherical Coordinates.** $x = \rho\sin(\phi)\cos(\theta)$; $y = \rho\sin(\phi)\sin(\theta)$; $z = \rho\cos(\phi)$

$$dV = \rho^2\sin(\phi) d\rho d\phi d\theta$$

- **General Coordinates in 2-d.**

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du dv$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

- **General Coordinates in 3-d.**

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du dv dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

- **Surface Area.** Area element of the portion of the graph $z = f(x,y)$ which lies over the rectangle $dxdy$

$$dA = \sqrt{1 + f_x^2 + f_y^2} dx dy$$