VECTOR FIELDS & FUNCTIONS (2-dims)

$$\{\text{Functions}\} \xrightarrow{\text{grad}} \left\{\begin{array}{c}\text{VECTOR}\\ \text{FIELDS}\end{array}\right\} \xrightarrow{\hat{k}\cdot\text{curl}} \{\text{FUNCTIONS}\}$$

$$f(x,y) \rightsquigarrow \nabla f = \langle f_x , f_y \rangle$$

$$\vec{F} = \langle P, Q \rangle \rightsquigarrow (Q_x - P_y)$$

$$\boxed{f(\vec{r}(b)) - f(\vec{r}(a)) = \int_C \nabla f \cdot d\vec{r}}$$

FUND. THM. OF PATH INTEGRALS

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \hat{k}\cdot \text{curl}(\vec{F})\, dA$$

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\, dA$$

GREEN'S THM

- $\hat{k}\cdot\text{curl}(\text{grad}(f)) = 0$ always
- If $\hat{k}\cdot\text{curl}(\vec{F}) = 0$, then $\vec{F}$ can be <u>locally</u> integrated to give a scalar f with $\nabla f = \vec{F}$.
- However $\vec{F} = \dfrac{\langle -y, x\rangle}{x^2+y^2}$ satisfies $\hat{k}\cdot\text{curl}(\vec{F}) = 0$

yet $\vec{F} \neq \nabla f$ for a globally defined function on $\mathbb{R}^2 - \{(0,0)\}$.

because $\oint_C \vec{F}\cdot d\vec{r} = 2\pi$ where $C: x^2+y^2 = 1$

VECTOR FIELDS & FUNCTIONS (3-dim)

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\{\text{Functions}\} \xrightarrow{\text{grad}} \left\{\begin{matrix}\text{Vector}\\ \text{Fields}\end{matrix}\right\} \xrightarrow{\text{curl}} \left\{\begin{matrix}\text{Vector}\\ \text{Fields}\end{matrix}\right\} \xrightarrow{\text{div}} \{\text{Functions}\}$$

$$f \rightsquigarrow \nabla f = \langle f_x, f_y, f_z \rangle$$

$$\vec{G} = \langle P, Q, R \rangle \rightsquigarrow \vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{F} = \langle P, Q, R \rangle \rightsquigarrow \nabla \cdot \vec{F} = P_x + Q_y + R_z$$

$$f(\vec{r}(b)) - f(\vec{r}(a)) = \int_C \nabla f \cdot d\vec{r}$$

FUND. THM. OF PATH INTEGRALS

$$\oint_{\partial S} \vec{G} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{G}) \cdot d\vec{S}$$

STOKES' THM.

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV$$

DIVERGENCE THM.

- $\nabla \times (\nabla f) = \vec{0}$ always
- $\nabla \cdot (\nabla \times \vec{G}) = 0$ always
- If $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ can be <u>locally</u> integrated to give f with $\nabla f = \vec{F}$.
- If $\nabla \cdot \vec{F} = 0$, then $\vec{F}$ can be <u>locally</u> integrated to give $\vec{G}$ with $\nabla \times \vec{G} = \vec{F}$.
- <u>NOT GLOBAL</u>: $\vec{F} = \dfrac{\langle -y, x, 0 \rangle}{x^2+y^2}$ satisfies $\nabla \times \vec{F} = \vec{0}$, yet $\oint_C \vec{F} \cdot d\vec{r} = 2\pi$
  $C: \langle \cos t, \sin t, 0 \rangle$, $0 \le t \le 2\pi$
- <u>NOT GLOBAL</u>: $\vec{F} = \dfrac{\langle x, y, z \rangle}{(x^2+y^2+z^2)^{3/2}}$ satisfies $\nabla \cdot \vec{F} = 0$,
  yet $\iint_S \vec{F} \cdot d\vec{S} = 4\pi$ $\quad S: x^2+y^2+z^2 = 1$