Intuition about the Laplacian

We shall see that, roughly speaking, the Laplacian of f measures the difference between the value of f at a point and the average values of f on small spherical neighborhoods of the point. We will make the rough intuition more precise as we go. For convenience we shall work around the origin $\mathbf{0} = (0, 0, 0)$, but the result holds at any point.

1. Let
$$g(t) \;=\; \iint_{S^2} f(t\mathbf{r})\, dS$$
We see that $g(t)$ measures the sum of $f(t\mathbf{r})$ over the unit sphere S^2 defined by $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$.

   Note the presence of t in the argument of f; as $\mathbf{r}$ ranges over S^2, the term $t\mathbf{r}$ ranges over the sphere of radius t about $\mathbf{0}$. Thus, the integral is essentially (up to a t^2 factor) summing the values of $f(\mathbf{r})$ over a sphere of radius t.

   Setting $t = 0$ we obtain
$$g(0) \;=\; \iint_{S^2} f(\mathbf{0})\, dS \;=\; 4\pi f(\mathbf{0})$$

2. Taylor's theorem gives
$$g(t) \;=\; g(0) \;+\; g'(0)t \;+\; \frac{g''(0)t^2}{2} \;+\; o\big(t^2\big)$$
where the notation $o(t^2)$ is used to denote the fact that the remainder tends to 0 faster than t^2; that is, $\frac{o(t^2)}{t^2} \to 0$ as $t \to 0$.

   We know $g(0)$ from item 1 above. We need to determine $g'(0)$ and $g''(0)$ next.

3. The derivative of g is obtained as follows
$$\begin{aligned}
\frac{dg}{dt} &= \frac{d}{dt}\iint_{S^2} f(t\mathbf{r})\, dS \\
&= \iint_{S^2} \frac{\partial f(t\mathbf{r})}{\partial t}\, dS \\
&= \iint_{S^2} \left[\frac{\partial f}{\partial x}(t\mathbf{r})\frac{\partial(tx)}{\partial t} \;+\; \frac{\partial f}{\partial y}(t\mathbf{r})\frac{\partial(ty)}{\partial t} \;+\; \frac{\partial f}{\partial z}(t\mathbf{r})\frac{\partial(tz)}{\partial t}\right] dS \\
&= \iint_{S^2} (\nabla f)|_{(t\mathbf{r})} \cdot \mathbf{r}\, dS \\
&= \iint_{S^2} (\nabla f)|_{(t\mathbf{r})} \cdot \hat{\mathbf{n}}\, dS \\
&= \iiint_{B^3} \nabla \cdot \big((\nabla f)|_{(t\mathbf{r})}\big)\, dV \\
&= \iiint_{B^3} t(\nabla^2 f)|_{(t\mathbf{r})}\, dV
\end{aligned}$$
The second equality is just taking a derivative inside of an integral. The $\frac{d}{dt}$ becomes $\frac{\partial}{\partial t}$ because the expression $f(t\mathbf{r})$ depends on $\mathbf{r} = (x, y, z)$ in addition to t, while outside the integral the

x, y, z variables have been "integrated" away and only t remains. Equality three is a result of the chain rule. Equality five is because, on the unit sphere, the radius vector $\mathbf{r}$ is identical to the unit outward pointing normal $\mathbf{n}$. Equality six is a result of the Divergence Theorem; the region B^3 is the solid unit ball $x^2 + y^2 + z^2 \leq 1$. The extra t term in equality seven is a result of the chain rule (when computing the divergence) again.

The t term inside the integral means that $g'(0) = 0$.

4. The second derivative of g is given by

$$\begin{aligned} g''(t) &= \frac{d}{dt} \iiint_{B^3} t(\nabla^2 f)|_{(t\mathbf{r})}\, dV \\ &= \iiint_{B^3} \frac{\partial}{\partial t} \left(t(\nabla^2 f)|_{(t\mathbf{r})} \right) dV \\ &= \iiint_{B^3} \left[(\nabla^2 f)|_{(t\mathbf{r})} + t \frac{\partial}{\partial t} \left((\nabla^2 f)|_{(t\mathbf{r})} \right) \right] dV \end{aligned}$$

Equality three is just the product rule for derivatives. The key fact is that the second term has a t factor, and will vanish at $t = 0$.

We obtain

$$g''(0) = \iiint_{B^3} \left[(\nabla^2 f)|_{(\mathbf{0})} + 0 \right] dV = \frac{4\pi}{3} (\nabla^2 f)|_{(\mathbf{0})}$$

5. Combining all these together into the Taylor expression in item 2 above gives

$$\iint_{S^2} f(t\mathbf{r})\, dS = 4\pi f(\mathbf{0}) + 0 + \frac{4\pi}{3} (\nabla^2 f)|_{(\mathbf{0})} \frac{t^2}{2} + o\big(t^2\big)$$

Solving for $(\nabla^2 f)|_{(\mathbf{0})}$ gives

$$(\nabla^2 f)|_{(\mathbf{0})} = \left[\iint_{S^2} f(t\mathbf{r})\, dS - 4\pi f(\mathbf{0}) \right] \frac{6}{4\pi t^2} + \frac{o(t^2)}{t^2}$$

Taking limits as $t \to 0$ gives

$$(\nabla^2 f)|_{(\mathbf{0})} = \lim_{t \to 0} \frac{6 \left[\frac{1}{4\pi} \iint_{S^2} f(t\mathbf{r})\, dS - f(\mathbf{0}) \right]}{t^2} \tag{1}$$

So the value of the Laplacian of f at a point measures the **difference** between the **average value** of f on a sphere of radius t about the point and the value of f at the point as $t \to 0$. This difference tends to 0 like t^2 and so the Laplacian measures the limit of the ratio of this and t^2. In the precise computations we found that there is a factor of 6 in this expression too.

Don't worry about the precise constants; just keep the intuition in your head. *The Laplacian is a type of second order derivative (hence the division by t^2 and the limit as $t \to 0$), and the numerator is the difference between the average value of f on a sphere of radius t about the point and the value of f at the point.*

6. **Remark 1.** The expression for the average value of f on the sphere of radius t used in equation (1) above may need some extra thought. Let S^2 denote the unit sphere, and let S^2_t

denote the sphere of radius t. We said that $\iint_{S^2} f(t\mathbf{r})\,dS$ was the same as the sum of f over the sphere S_t^2 up to a factor of t^2. You should check that indeed

$$t^2 \iint_{S^2} f(t\mathbf{r})\,dS \;=\; \iint_{S_t^2} f(\mathbf{r})\,dS$$

Now, the average value of f on S_t^2 is

$$\frac{\iint_{S_t^2} f(\mathbf{r})\,dS}{4\pi t^2} \;=\; \frac{t^2 \iint_{S^2} f(t\mathbf{r})\,dS}{4\pi t^2} \;=\; \frac{\iint_{S^2} f(t\mathbf{r})\,dS}{4\pi}$$

This last term is what we used for the average of f on S_t^2 in equation (1).

7. **Remark 2.** The expression (1) for the Laplacian can be derived in any dimension (not just 3–d). In general, one obtains

$$(\nabla^2 f)|_{(\mathbf{0})} \;=\; \lim_{t\to 0} \frac{2n\left[\frac{1}{V_{n-1}(S^{n-1})}\int_{S^{n-1}} f(t\mathbf{r})\,dV - f(\mathbf{0})\right]}{t^2}$$

Here the integral is over the unit $(n-1)$–dimensional sphere S^{n-1} (defined by the equation $x_1^2 + \cdots + x_n^2 = 1$) in $\mathbb{R}^n$, and $V_{n-1}(S^{n-1})$ denotes the $(n-1)$–dimensional volume of S^{n-1}.

Notation. It is cleaner to use the single integral notation $\int_{E^m}$ to denote an m-dimensional integral, rather than something unwieldy such as $\int\cdots\int_{E^m}$.

8. **Remark 3.** Here is an instance of the Laplacian in "real life." The Heat Equation is given by

$$\frac{\partial u}{\partial t} \;=\; \alpha \nabla^2 u$$

Here $u = u(x, y, z, t)$ denotes the temperature at the point (x, y, z) of some object at time t. The constant $\alpha > 0$ (called the *thermal diffusivity* of the material) depends on the physical properties of the object in question.

For example, suppose that at a given time t, the average temperatures of points on small spheres around (x, y, z) are greater than the temperature at (x, y, z). Then from the intuitive interpretation of ∇^2 above we have that $\nabla^2 u > 0$ at the point. Furthermore, we expect that $\frac{\partial u}{\partial t} > 0$ at the point because the warmer neighbors are "heating the point up." Plugging a positive term $\nabla^2 u$ into the heat equation we obtain $\frac{\partial u}{\partial t} > 0$, which agrees with our intuition.

Note that for a "steady state" solution (i.e., solution does not change in time) to the heat equation, we have $\frac{\partial u}{\partial t} = 0$ and so $\nabla^2 u = 0$ everywhere. This means that in a steady state situation, the temperature at any point is equal to the average of the temperatures of neighboring points (on small spheres about the point). This "average value property" is a good intuition to have about steady state heat distributions.

9. **Remark 4.** The Laplacian in one dimension is just the second derivative, $\nabla^2 f = \frac{d^2 f}{dx^2} = f''(x)$. It is a good exercise to check that indeed $f''(x) = \lim_{h\to 0} 2[(f(x+h) + f(x-h))/2 - f(x)]/h^2$. Also, note that the only functions which satisfy $f''(x) = 0$ are the straight line functions $f(x) = ax + b$ which clearly satisfy the average value property: $(f(x+h) + f(x-h))/2 = f(x)$.

10. **Remark 5.** The derivation of the Heat Equation above from physical principles is a good example of how the Divergence Theorem is used in setting up mathematical models in your physics or engineering courses.

    The basic strategy is usually to formulate an integral version of the physical principle, and then to convert to a differential formulation. The integral formulation may involve double and triple integrals, and we use the Divergence Theorem to convert all to triple integrals, and then compare integrands.

    Using the notation above, we let $u(x, y, z, t)$ denote the temperature at the point (x, y, z) of a body at time t.

    - Since heat "flows" from points with higher temperatures to points with lower temperatures whereas ∇u points in the direction of increasing temperature, it makes sense to define the heat flux to be proportional to $-\nabla u$. So the heat flux is defined to be

      $$-\lambda \nabla u$$

      where we suppose for simplicity that λ is a constant throughout the body. This quantity λ is called the *thermal conductivity* of the material.
    - The net heat flux (heat energy flow in unit time) out of a 3-dimensional region E of the body is given by

      $$\iint_{\partial E} -\lambda \nabla u \cdot d\mathbf{S}$$

      where ∂E denotes the (closed surface) boundary of the region E. This is known as *Fourier's Law.*

      By the Divergence Theorem, this can be rewritten as a volume integral

      $$\iiint_E \nabla \cdot (-\lambda \nabla u)\, dV \;=\; \iiint_E -\lambda \nabla^2 u\, dV$$
    - On the other hand, the net heat energy contained in the region E of the body can be written as a volume integral

      $$\iiint_E \rho \sigma u\, dV$$

      where ρ is the *mass density*, and σ is the *specific heat capacity* of the material in the body.
    - The net heat flux across ∂E is the net flow of heat energy out of the region E in unit time, which is equal to the negative of the rate of change of heat energy in the region E (the underlying physical principle here is *conservation of energy*)

      $$\iiint_E -\lambda \nabla^2 u\, dV \;=\; -\frac{\partial}{\partial t}\left(\iiint_E \rho \sigma u\, dV\right) \;=\; -\iiint_E \frac{\partial}{\partial t}(\rho \sigma u)\, dV$$
    - This integral equation holds true for **any** region E in the body. In particular, it holds for regions E_r which are balls of radius $r \to 0$ about any point P. Therefore, we can compare integrands to obtain

      $$\rho \sigma \frac{\partial u}{\partial t} \;=\; \lambda \nabla^2 u$$

      at all points P of the body.

      This is the heat equation in Remark 3 above, with $\alpha = \frac{\lambda}{\rho \sigma}$.