§**9.1 Questions 10,18,20,28,30;** §**9.2 Questions 4,8,24,30,32**
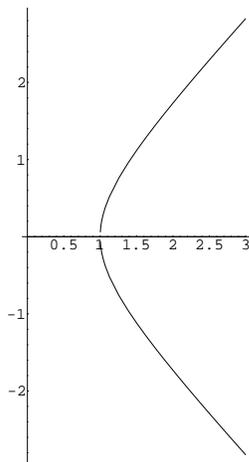
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10. Note that the standard trigonometric identities show that $x^2 - y^2 = 1$. Furthermore, as θ varies in the interval $(-\pi/2, \pi/2)$, we see that x takes on all values greater than or equal to one and y takes on all real values. So the curve should be one branch of the hyperbola $x^2 - y^2 = 1$:

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18. Notice that $x - 2 = \cos t$ and $y - 3 = \sin t$. It follows that, as t varies from 0 to 2π, the point describes a single counterclockwise loop of the circle $(x-2)^2 + (y-3)^2 = 1$. The point starts and ends at $(3, 3)$. This circle has center $(2, 3)$ and radius 1.

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20. Clearly the point lies on the parabola $x = y^2$. We notice also that $0 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$. As t varies from 0 to 4π, the point moves along this parabola from the point $(1, 1)$ to $(1, -1)$ and back twice.

28.

a). We have $x = 250t\sqrt{3}$, $y = 250t - 4.9t^2 = t(250 - 4.9t)$. The bullet hits the ground when $y = 0$. This happens when $t = 250/4.9 \simeq 51s$. At this time $x = 250^2\sqrt{3}/4.9 \simeq 22{,}092m$, so this is the distance from the gun when the bullet hits the ground.

b). The maximum height occurs when y is maximum. This occurs when $t = 125/4.9$. Consequently, the maximum height is
$125(250 - 125)/4.9 = 125^2/4.9 \simeq 3{,}189m$.

c). Notice that $t = x/(250\sqrt{3})$. Consequently

$$y = \frac{x}{\sqrt{3}} - \frac{4.9x^2}{3(250)^2},$$

which is the equation of a parabola.

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30. The parametric equations are derived in exactly the way we did it in class for the cycloid. Rather than draw the curve for you, I will give you a link to an animation of the construction.

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4. We have $x'(t) = \sin t + t\cos t$, $y'(t) = \cos t - t\sin t$. Consequently, at $t = \pi$, we have $x = 0$, $y = -\pi$, $x' = -\pi$, $y' = -1$. It follows that the equation of the tangent is $y = x/\pi - \pi$.

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8. We note that this curve passes through the origin exactly when $t = n\pi$ for $n = 0, \pm 1, \pm 2, \dots$. Differentiation gives $x'(t) = \cos t$, $y'(t) = \left[\cos(t + \sin t)\right](1 + \cos t)$. Consequently, at $t = n\pi$, we have $x' = (-1)^n$, $y' = (-1)^n\left(1 + (-1)^n\right) = (-1)^n + 1$. Notice therefore that $dy/dx = 0$ if n is odd and $dy/dx = 2$ if n is even. It follows that there two tangents with equations $y = 0$, $y = 2x$.

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24. Notice that $x(t) = -\cos 2t$ and $y(t) = -\tan t \cos 2t$. For a self intersection, we require two values t_0, t_1 such that $x(t_0) = x(t_1)$ and $y(t_0) = y(t_1)$. This requires either $\cos 2t_0 = \cos 2t_1$ and $\tan t_0 = \tan t_1$ or $\cos 2t_0 = \cos 2t_1 = 0$. The first condition cannot be satisfied, because of the nature of the cosine and tangent functions. However, the second equation is satisfied on putting $t_0 = -\pi/4$ and $t_1 = \pi/4$. These are the two parameter values at which the curve passes through the origin. The slope of the curve is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\tan t \sin t - \sec^2 t \cos 2t}{2 \sin t}.$$

At t_0, t_1 the slope is therefore ± 1. The tangents are $y = \pm x$.

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30. Notice that $x + y = 2t$ and so the equation is $x = (x + y)/2 - 2/(x + y)$. This reduces to $y^2 - x^2 = 4$ which is the equation of a hyperbola. Positive values of t yield the upper branch of the hyperbola. The curve crosses the line $y = 5/2$ when $t = 1/2$, 2. Consequently the required area is

$$A = \int_{1/2}^{2} \left(\frac{5}{2} - t - \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) dt = \int_{1/2}^{2} \left(\frac{5}{2} + \frac{5}{2t^2} - t - \frac{1}{t} - \frac{1}{t^3}\right) dt$$

$$= \left[\frac{5t}{2} - \frac{5}{2t} - \frac{t^2}{2} - \ln 2 + \frac{1}{2t^2}\right]_{1/2}^{2} = \frac{15}{4} - 2\ln 2.$$

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32. The parameter interval $[0, 2\pi]$ yields a single copy of the asteroid. Using the symmetry of this curve, we see that its area is given by

$$A = 4 \int_0^{\pi/2} a \sin^3 \theta \, 3a \cos^2 \theta \sin \theta \, d\theta = 12a^2 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta.$$

Using your favourite trigonometric identities, you obtain

$$\sin^4 \theta = \frac{1}{4}(1 - \cos 2\theta)^2 = \frac{1}{4} - \frac{1}{2}\cos\theta + \frac{1}{4}\cos^2 2\theta$$

$$= \frac{1}{4} - \frac{1}{2}\cos\theta + \frac{1}{8}(1 + \cos 4\theta) = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

So

$$2\sin^4 \theta \cos^2 \theta = (\frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta)(1 + \cos 2\theta)$$

$$= \frac{1}{8} - \frac{1}{16}\cos 2\theta - \frac{1}{8}\cos 4\theta + \frac{1}{16}\cos 6\theta.$$

Although this seems quite formidable, note that the integral below shows that only the first term above is of any significance, the others integrate to zero. So you are only require to take care over the first term. The required area is consequently

$$A = 6a^2 \int_0^{\pi/2} (\frac{1}{8} - \frac{1}{16}\cos 2\theta - \frac{1}{8}\cos 4\theta + \frac{1}{16}\cos 6\theta) \, d\theta = 6a^2 \frac{1}{8}\frac{\pi}{2} = \frac{3\pi a^2}{8}.$$