CALCULUS III FALL 1999
HOMEWORK 2 – ANSWERS

§9.3 Questions 6,8,12,22,24; §9.4 Questions 26,36,40,42,78

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6. We have

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$$

and

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta.$$

So the required length is

$$L = \int_0^\pi \sqrt{a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta}\,d\theta = \int_0^\pi |a|\theta\,d\theta = \frac{1}{2}\pi^2|a|.$$

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8. We have

$$\frac{dx}{dt} = e^t - 1 \qquad \text{and} \qquad \frac{dy}{dt} = 2e^{t/2}.$$

So the required length is

$$L = \int_0^1 \sqrt{(e^t-1)^2 + 4e^t}\,dt = \int_0^1 (1+e^t)\,dt = e.$$

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12. We have

$$\frac{dx}{d\theta} = -\frac{2a}{\sin^2\theta} \qquad \text{and} \qquad \frac{dy}{d\theta} = 4a\sin\theta\cos\theta.$$

So the required length is

$$L = \int_{\pi/4}^{\pi/2} \sqrt{\frac{4a^2}{\sin^4\theta} + 16a^2\sin^2\theta\cos^2\theta}\,d\theta \simeq 2.26|a|.$$

22. We have $dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so the required surface area is

$$S = 2\pi \int_0^1 3t^2 \sqrt{(3-3t^2)^2 + 36t^2}\, dt = 6\pi \int_0^1 t^2 \sqrt{9t^4 + 18t^2 + 9}\, dt$$
$$= 6\pi \int_0^1 t^2(3t^2+3)\, dt = 18\pi \Big[\frac{t^5}{5} + \frac{t^3}{3}\Big]_0^1 = \frac{48}{5}\pi.$$

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24. Notice that, as θ varies from $-\pi$ to π the curve is below the x-axis and as it varies from 0 to π it desrcibes the mirror image above the x-axis, this follows from the fact that sin is an odd function. Note that for $0 \leqslant \theta \leqslant \pi$, $dy/dx = (\cos\theta - \cos 2\theta)/(\sin 2\theta - \sin\theta)$ except when $\theta = 0,\ \pi/3,\ \pi$. In this parameter interval, $dy/d\theta > 0$ for $\theta < \theta_0$ where $\cos\theta_0 = (1-\sqrt{5})/4$ and $dy/d\theta < 0$ for $\theta > \theta_0$. Similarly, $dx/d\theta > 0$ for $\theta < \pi/3$; and $dx/d\theta < 0$ for $\theta > \pi/3$. So $dy/dx > 0$ if $0 < \theta < \pi/3$; $dy/dx < 0$ if $\pi/3 < \theta < \theta_0$; $dy/dx > 0$ if $\theta_0 < \theta < \pi$. From these observations we can deduce that the curve has no self intersections for the parameter interval $[0,\pi]$. This can be much more easily observed by asking your calculator to plot the curve. But you should at least know how to derive it without a calculator. The required surface area is obtained by rotating the part of the curve corresponding to $0 < \theta < \pi$ about the x-axis. This gives

$$S = 2\pi \int_0^\pi (2\sin\theta - \sin 2\theta)\sqrt{4(\sin\theta - \sin 2\theta)^2 + 4(\cos\theta - \cos 2\theta)^2}\, d\theta$$
$$= 8\pi\sqrt{2} \int_0^\pi (1-\cos\theta)\sin\theta\sqrt{1 - \sin\theta\sin 2\theta - \cos\theta\cos 2\theta}\, d\theta$$
$$= 8\pi\sqrt{2} \int_0^\pi (1-\cos\theta)^{3/2}\sin\theta\, d\theta = 8\pi\sqrt{2}\big[2(1-\cos\theta)^{5/2}/5\big]_0^\pi = \frac{128}{5}\pi.$$

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26. We have $x = r\cos\theta = 2\sin\theta\cos\theta = \sin 2\theta$ and $y = r\cos\theta = 2\sin^2\theta = 1 - \cos 2\theta$. Consequently, $x^2 + (y-1)^2 = 1$; which is the circle center $(0,1)$ and radius 1.

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36. We have $r^2\cos^2\theta - r^2\sin^2\theta = 1$ which is equivalent to $r^2\cos^2 2\theta = 1$ or $r^2 = \sec 2\theta$ for $-\pi/4 < \theta < \pi/4$.

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40. We have $x = r\cos\theta = -4\sin\theta\cos\theta = -2\sin 2\theta$ and $y = r\sin\theta = -4\sin^2\theta = 2(\cos 2\theta - 1)$. Consequently, $x^2 + (y+2)^2 = 4$ and therefore the curve is the circle center $(0,-2)$ and radius 2. Alternatively, we recall from class that the curve $r = -4\cos\theta$ is the circle center $(-2,0)$ and radius 2. Now note that $\sin\theta = \cos(\theta+\pi/2)$ and therefore the required cirve arises from rotating the latter circle through $\pi/2$.

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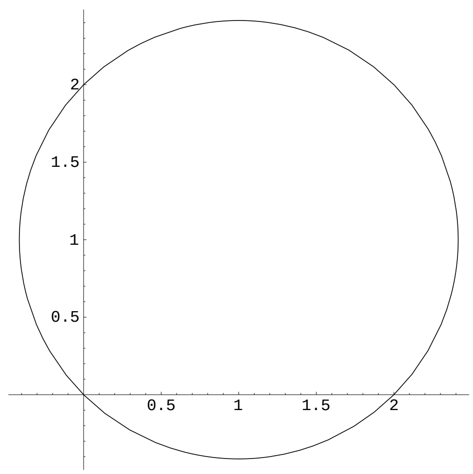
42. Note that $r = 2(\sin\theta + \cos\theta) = 2\sqrt{2}\cos(\theta - \pi/4)$. So we should expect the curve obtained by rotating $r = 2\sqrt{2}\cos\theta$ through $\pi/4$ (counterclockwise) about the origin. The latter is a circle and so we see that the required curve is



and this curve is described once as θ varies from 0 to π.

78.

a). $r = \sin(\theta/2)$ must be a curve which stays within distance one of the origin. It will pass through the origin when $\theta = 0,\ 2\pi,\ 4\pi$. Other points where it crosses the y-axis arise when $\cos\theta = 0$; these are

$$\theta = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2.$$

Calculating the y-coordinate for these values shows that, apart from the origin, the curve crosses the y-axis twice more. The only possible curve is VI.

b). It will pass through the origin when $\theta = 0,\ 4\pi,\ 8\pi$, so again we have two passes through the origin. Other intersections with the y-axis come from

$$\theta = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2,\ 9\pi/2,\ 11\pi/2,\ 13\pi/2,\ 15\pi/2.$$

This leads to four different values of y, namely $\pm\sin\pi/8,\ \pm\sin 3\pi/8$, so there are four crossings of the y-axis different from the origin. This now has to be curve III.

c). In this case, we always have $|r| \geqslant 1$, so the only possibility is IV.

d). This curve passes through the origin every time θ is a multiple of π. There are no other places where it crosses the x-axis, so it must be V.

e). This crosses the x-axis at all multiples of π. At the odd multiples $r = -3$ and at the even multiples $r = 5$. So the curve crosses the x-axis at the points $(3, 0)$ and $(5, 0)$ (and the origin). The only possibility is II.

f). As θ approaches zero this r approaches ∞; and as θ approaches ∞, r approaches zero from above. So this curve is a spiral and must be I.