CALCULUS III FALL 1999
HOMEWORK 6 – ANSWERS

§10.2 Questions 14,20,26,58; §10.3 Questions 8,10,20,26

14.

$$\sum_{n=1}^{\infty} \frac{1}{e^{2n}} = \sum_{n=0}^{\infty} \frac{1}{e^{2n}} - 1 = \frac{1}{1-1/e^2} - 1 = \frac{1}{e^2-1}.$$

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20. Note that $n^2/(3(n+1)(n+2)) \to 1/3$ as $n \to \infty$ and so the series diverges.

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26. The partial sums are given by

$$\begin{aligned} s_N &= \sum_{n=1}^{N} \frac{1}{4n^2-1} = \frac{1}{2}\sum_{n=1}^{N}\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big) \\ &= \frac{1}{2}\Big[\Big(1-\frac{1}{3}\Big) + \Big(\frac{1}{3}-\frac{1}{5}\Big) + \cdots + \Big(\frac{1}{2N-1} - \frac{1}{2N+1}\Big)\Big] = \frac{1}{2}\Big[1 - \frac{1}{2N+1}\Big] \to \frac{1}{2} \end{aligned}$$

as $N \to \infty$. It follows that $\sum_{n=1}^{\infty} 1/(4n^2-1) = 1/2$.

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58. Note that $CD = b\sin\theta,\ DE = CD\sin\theta$ and so on. The required sum is therefore

$$b\sum_{n=1}^{\infty} \sin^n\theta = b\Big(\frac{1}{1-\sin\theta} - 1\Big) = \frac{b\sin\theta}{1-\sin\theta}.$$

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8. The function $f(x) = 1/(x^2-1)$ is clearly positive and decreasing on the interval $[2,\infty)$, so we will apply the integral test.

$$\int_2^{\infty} \frac{dx}{x^2-1} = \frac{1}{2}\lim_{A\to\infty}\int_2^{A}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)\,dx = \frac{1}{2}\lim_{A\to\infty}\Big[\ln\frac{A-1}{A+1} - \ln\frac{1}{3}\Big] \to \frac{1}{2}\ln 3,$$

as $A \to \infty$. It follows that the series converges.

10. We put $f(x) = x/2^x$ and note that $f'(x) = (1 - x\ln 2)/2^x$. Consequently f is positive and decreasing on $[2, \infty)$, so we may apply the integral test.

$$\begin{aligned}\int_2^\infty \frac{x\,dx}{2^x} &= \lim_{A\to\infty}\left(\left[-\frac{x}{2^x\ln 2}\right]_2^A + \frac{1}{\ln 2}\int_2^A 2^{-x}dx\right)\\ &= \frac{1}{2}\left(\frac{1}{\ln 2} + \frac{1}{(\ln 2)^2}\right) \qquad \text{because } \lim_{A\to\infty}\frac{A}{2^A} = 0.\end{aligned}$$

So the series converges.

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20. The function $f(x) = 1/(x\ln x(\ln(\ln x))^p)$ is clearly positive and decreasing on $[3, \infty)$ if $p > 0$. If $p \neq 1$, integration gives

$$\int_3^\infty \frac{dx}{x\ln x(\ln(\ln x))^p} = \lim_{A\to\infty}\left[\frac{(\ln\ln x)^{-p+1}}{-p+1}\right]_3^A.$$

This limit exists precisely when $p > 1$. In case $p = 1$, we have

$$\int_3^\infty \frac{dx}{x\ln x\ln(\ln x)} = \lim_{A\to\infty}\Big[\ln\ln x\Big]_3^A,$$

which is infinite. If $p \leqslant 0$, the terms of the series do not converge to 0. Putting all this together shows that the series converges precisely when $p > 1$.

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26. In the usual notation, we have

$$R_n = \sum_{k=n+1}^\infty \frac{1}{k^5} < \int_n^\infty x^{-5}\,dx = \lim_{A\to\infty}\left[-\frac{1}{4x^4}\right]_n^A = \frac{1}{4n^4}.$$

So we choose n such that $1/4n^4 < 0.0001$. This requires $n \geqslant 8$. Consequently s_8 is a sufficiently good estimate of the sum. We have $s_8 \simeq 1.03688\ldots$ and so $\sum_{n=1}^\infty 1/n^5 = 1.037$ to three places of decimals.