CALCULUS III FALL 1999
HOMEWORK 7 – ANSWERS

§10.5 Questions 6,14,18,24,28; §10.6 Questions 4,10,12,18

---

6. Note that $1/\sqrt{n+3}$ decreases to zero as n increases to infinity. So the alternating series test shows that this series is convergent.

---

14. If $f(x) = (\ln x)/x$ then $f'(x) = (1 - \ln x)/x^2 < 0$ if $x > e$. Also L'Hôpital's rule gives $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1/x = 0$. Putting these two observations together shows that $(\ln n)/n$ decreases to zero as n increases to infinity. The alternating series test now shows that this series converges.

---

18. Note that $\cos \pi/n \to 1$ as $n \to \infty$, so the terms of the series do not approach zero. The series must therefore diverge.

---

24. We put $f(x) = (\ln x)^p/x$ and note that $f'(x) = (\ln x)^{p-1}(p - \ln x)$. Differentiation gives $f'(x) = (\ln x)^{p-1}(p - \ln x)/x^2$ and so f decreases on (e^p, ∞). Using L'Hôpital's rule repeatedly, we also see that

$$\lim_{x\to\infty} f(x) = p \lim_{x\to\infty} \frac{(\ln x)^{p-1}}{x} = \cdots = p(p-1)\ldots(p-m) \lim_{x\to\infty} \frac{(\ln x)^{p-m-1}}{x} = 0,$$

where m is the least integer which exceeds p. So the alternating series test shows that the series converges.

---

28. Notice that, if $f(x) = x/4^x$ then $f'(x) = (1 - x\ln 4)/4^x$ and so f decreases on $(1/\ln 4, \infty)$. L'Hôpital's rule gives $\lim_{x\to\infty} x/4^x = \lim_{x\to\infty} 1/(4^x \ln 4) = 0$. The alternating series test now shows that the series converges. In order to obtain the required estimate, we must evaluate the partial sum s_n where $n/4^n < 0.002$. This requires $4^n/n > 500$ which is true for $n = 6$. So the required estimate is given by

$$s_6 = -\frac{1}{4} + \frac{2}{16} - \frac{3}{64} + \frac{4}{256} - \frac{5}{1024} = -\frac{165}{1024} \simeq -0.16113.$$

4. The series is absolutely convergent, since the ratio test gives

$$\frac{3^{n+1}}{(n+1)!}\frac{n!}{3^n} = \frac{3}{n+1} \to 0.$$

---

10. The corresponding absolute series is $\sum_{n=1}^{\infty} \sqrt{n}/(n+1)$. We compare this with $\sum_{n=1}^{\infty} 1/\sqrt{n}$. The limit form of the comparison test gives $n/(n+1) \to 1$ and so our series is not absolutely convergent. We now check on the convergence using the alternating series test. We put $f(x) = \sqrt{x}/(x+1)$ and note that $f'(x) = (1-x)/(2\sqrt{x}(x+1)^2) < 0$, if $x > 1$. So the alternating series test shows that this series converges conditionally.

---

12. Notice that $2^n/(n^2+1) \to \infty$, so the terms of our series do not approach zero. Consequently the series diverges.

---

18. Note that

$$\left|\frac{\cos(n\pi/6)}{n\sqrt{n}}\right| \leqslant \frac{1}{n^{3/2}},$$

and so the comparison test shows that our series is absolutely convergent.