CALCULUS III FALL 1999
HOMEWORK 8 – ANSWERS

§10.8 Questions 6,14,22,28; §10.9 Questions 2,10,20,28,36

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6. The ratio test gives

$$\frac{|x|^{n+1}}{(n+1)^2}\frac{n^2}{|x|^n} = \frac{n^2|x|}{(n+1)^2} \to |x|.$$

So the radius of convergence is 1. At $x = \pm 1$ the corresponding absolute series is $\sum_{n=1}^{\infty} 1/n^2$ which converges. So the interval of convergence is $[-1, 1]$.

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14. The ratio test gives

$$\frac{\sqrt{n+1}|3x+2|^{n+1}}{\sqrt{n}|3x+2|^n} \to |3x+2|.$$

So this series converges when $|3x+2| < 1$ which means $|x+2/3| < 1/3$. So the radius of convergence is $1/3$. At $x = -1$ or $-1/3$ the terms of the series do not approach zero. So the interval of convergence is $(-1, -1/3)$.

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22. The ratio test gives

$$\frac{|x+1|^{n+1}}{(n+1)(n+2)}\frac{n(n+1)}{|x+1|^n} \to |x+1|.$$

So the radius of convergence is 1. At $x = -2,\ 0$ the corresponding absolute series is $\sum_{n=1}^{\infty} 1/(n(n+1))$ which is convergent (by comparison with $\sum_{n=1}^{\infty} 1/n^2$). So the interval of convergence is $[-2, 0]$.

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28. The ratio test gives

$$\frac{2.4.6\ldots(2n+2)|x|}{1.3.5\ldots(2n+1)}\,\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} = \frac{(2n+2)|x|}{2n+1} \to |x|.$$

So the radius of convergence is one. Notice that

$$\frac{2.4.6.\ldots(2n)}{1.3.5\ldots(2n-1)} = \frac{2}{1}\frac{4}{3}\frac{6}{5}\cdots\frac{2n}{2n-1} > 1 \qquad \text{for all } n.$$

So, at $x = \pm 1$, the terms of the series do not converge to zero. It follows that the series diverges at $x = \pm 1$ and so the interval of converegnce is $(-1, 1)$.

2.

$$\frac{x}{1-x} = x\frac{1}{1-x} = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1},$$

with radius of convergence 1. The series clearly diverges at $x = \pm 1$, so the interval of convergence is $(-1, 1)$.

10.

$$\begin{aligned} f(x) &= \frac{x}{x^2-3x+2} = \frac{x}{x-2} - \frac{x}{x-1} = \frac{x}{1-x} - \frac{x}{2(1-x/2)} \\ &= \sum_{n=0}^{\infty} x^{n+1} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n} \end{aligned}$$

these series have respective radii of convergence 1 and 2

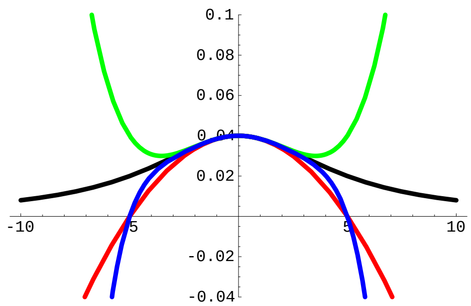
$$= \sum_{n=0}^{\infty}\Big(1 - \frac{1}{2^{n+1}}\Big)x^{n+1}.$$

We require both the above series to converge and so the overall radius of convergence is 1. At $x = \pm 1$ the terms of the series do not converge to zero and so the interval of convergence is $(-1, 1)$.

20.

$$f(x) = \frac{1}{x^2+25} = \frac{1}{25\big(1-(-(x/5)^2)\big)} = \frac{1}{25}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{5^{2n}} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{5^{2n+2}},$$

which has interval of convergence $(-5,5)$. The first few Maclaurin polynomials are graphed below. Notice that, as n increases, the polynomials provide increasingly better aproximations to $f(x)$. The function f is shown in black.



The degree 2, 4, 6 approximations have colours red, green blue respectively.

28.

$$\frac{1}{1+x^6} = \frac{1}{1-(-x^6)} = \sum_{n=0}^{\infty}(-1)^n x^{6n}.$$

Consequently

$$\int_0^{1/2}\frac{dx}{1+x^6} = \Big[\sum_{n=0}^{\infty}\frac{(-1)^n}{6n+1}x^{6n+1}\Big]_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(6n+1)2^{6n+1}}.$$

This is an alternating series of terms decreasing to zero. So, in order to get the required accuracy, we choose n so that $(6n+1)2^{6n+1} > 10^6$. This is satisfied when $n=3$. So the required estimate is given by

$$\int_0^{1/2}\frac{dx}{1+x^6} = \frac{1}{2} - \frac{1}{7.2^7} + \frac{1}{13.2^{13}} \simeq 0.498893.$$

36. This question is based on the fact that $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ if $|x| < 1$.

(a)

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \qquad \text{if } |x| < 1.$$

(b)

(i)

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2} \qquad \text{if } |x| < 1.$$

(ii) The above series converges when $x = 1/2$ and so $\sum_{n=1}^{\infty} n/2^n = 2$.

(c)

(i)

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\frac{d}{dx}\sum_{n=1}^{\infty} nx^{n-1}$$

$$= x^2\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2x^2}{(1-x)^3} \qquad \text{if } |x| < 1.$$

(ii) The above series converges when $x = 1/2$ and so $\sum_{n=2}^{\infty}(n^2-n)/2^n = 4$.

(iii)

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4 + 2 = 6.$$