CALCULUS III FALL 1999

HOMEWORK 9 – ANSWERS

§10.10 Questions 8,12,24,34,38,44

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8. Note that

$$f^{(4n)}(-\pi/4) = \cos(-\pi/4) = 1/\sqrt{2}, \qquad f^{(4n+1)}(-\pi/4) = -\sin(-\pi/4) = 1/\sqrt{2},$$
$$f^{(4n+2)}(-\pi/4) = -\cos(-\pi/4) = -1/\sqrt{2}, \quad f^{(4n+3)}(-\pi/4) = \sin(-\pi/4) = -1/\sqrt{2}.$$

It follows that

$$\cos x = \frac{1}{\sqrt{2}} + \frac{x+\pi/4}{\sqrt{2}} - \frac{(x+\pi/4)^2}{2\sqrt{2}} - \frac{(x+\pi/4)^3}{6\sqrt{2}} + \cdots$$

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12. We have $f'(x) = 1/x$, $f''(x) = -1/x^2$, in general

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \qquad \text{and so} \qquad f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n}.$$

Hence

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n.$$

24.

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\Big(1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\Big) = \frac{1}{2}\Big(1 + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}\Big).$$

34.

$$\int \frac{\sin x}{x}\,dx = \int \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} + c.$$

38.

$$\int_0^{1/2}\cos x^2\,dx = \int_0^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n}}{(2n)!}\,dx = \int_0^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}\,dx$$
$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\right]_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)(2n)!2^{4n+1}}.$$

This is an alternating series of terms decreasing to zero. So, in order to get the required accuracy, we choose n so that $(4n+1)(2n)!2^{4n+1} > 10^3$. This is satisfied when $n = 2$. So the required estimate is given by

$$\int_0^{1/2}\cos(x^2)\,dx = \frac{1}{2} - \frac{1}{10(2^5)} \simeq 0.497.$$

44.

$$e^x \ln(1-x) = \Big(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\Big)\Big(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\Big)$$
$$= -x - \frac{3x^2}{2} - \frac{x^3}{3} - \frac{x^3}{2} - \frac{x^3}{2} - \cdots = -x - \frac{3x^2}{2} - \frac{4x^3}{3} - \cdots.$$

All subsequent terms involve x^4 and higher powers.