CALCULUS III FALL 1999

TEST 1

In order to get full credit, all answers must be accompanied by appropriate justifications.

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1. Find the equation of the tangent line to the curve

$$x(t) = t^2 + 2t \qquad y(t) = t^3 - t,$$

at the point $(3, 0)$.

(15 points)

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$x'(t) = t^2 + 2t$, $y'(t) = 3t^2 - 1$. Consequently $dy/dx = (3t^2 - 1)/(2t + 2)$. The point $(3, 0)$ corresponds to $t = 1$ and so $dy/dx = 1/2$ at this point. The equation of the tangent is therefore $y = (x - 3)/2$.

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2. Find the total length of the astroid

$$x(t) = a\cos^3 t \qquad y(t) = a\sin^3 t \qquad \text{where } a > 0.$$

(15 points)

....

The astroid is described once as t varies from 0 to 2π. We have $x'(t) = -3a\cos^2 t \sin t$ and $y'(t) = 3a\sin^2 t \cos t$. Consequently the required length is

$$L = \int_0^{2\pi} \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\, dt$$

$$= 3a \int_0^{2\pi} |\cos t \sin t|\, dt = 6a \int_0^{\pi/2} \sin 2t\, dt = 6a.$$

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3. Find the polar coordinates of the points of intersection of the curves which satisfy the polar equations

$$r^2 = 4\cos\theta \qquad \text{and} \qquad r = 1 - \cos\theta.$$

(20 points)

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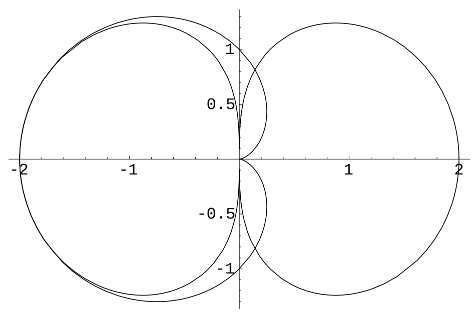
We set $4\cos\theta = (1-\cos\theta)^2$. Thus $\cos^2\theta - 6\cos\theta + 1 = 0$. Consequently $\cos\theta = 3\pm 2\sqrt{2}$, of course the only possibility is $\cos\theta = 3-2\sqrt{2}$. Put $\theta_0 = \cos^{-1}(3-2\sqrt{2}) \simeq 1.398$ radians. When $\theta = \theta_0$, $r = 2\sqrt{2}-2$. Consequently, this gives two points of intersection $(2\sqrt{2}-2, \theta_0)$, $(2\sqrt{2}-2, 2\pi-\theta_0)$. Graphing the curves gives



We see that there are two more points of intersection, they are the origin and the point with polar coordinates $(2,\pi) = (-2,0)$.

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4. Find the area enclosed by the inner loop of the lemniscate $r = 1 - 3\sin\theta$.

(25 points)

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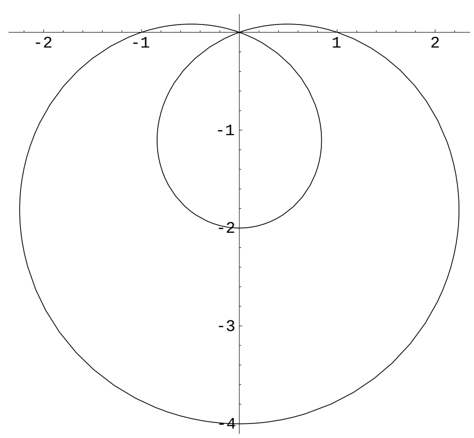
The lemniscate is shown below



It passes through the origin when $\sin\theta = 1/3$. There is one such value of θ in the interval $(0,\pi/2)$, we call it θ_0. By symmetry, the required area is

$$A = \int_{\theta_0}^{\pi/2} (1-3\sin\theta)^2\, d\theta = \int_{\theta_0}^{\pi/2} \left(\frac{11}{2} - 6\sin\theta - \frac{9}{2}\cos 2\theta\right)\, d\theta$$

$$= \frac{11\pi}{4} - \left(\frac{11\theta_0}{2} + 6\cos\theta_0 - \frac{9}{2}\sin\theta_0\cos\theta_0\right) = \frac{11\pi}{4} - \frac{11}{2}\sin^{-1}\frac{1}{3} - 3\sqrt{2} \simeq 2.528.$$

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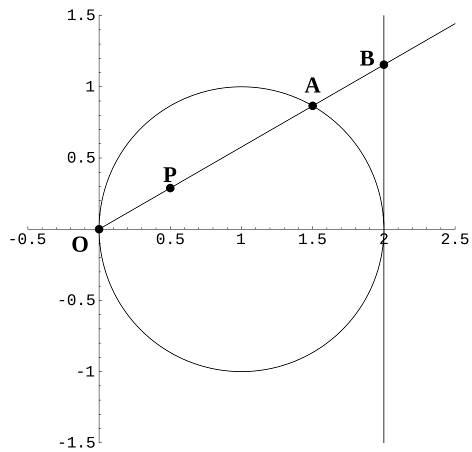
5. In the diagram below, the point B moves along the vertical line $x = 2$, the point A denotes the intersection of the line joining the origin O to B with the circle center $(1, 0)$ and radius 1. The point P is chosen, on the line joining O and B, so that the distance OP equals the distance AB. The point P describes the Cissoid of Diocles. Find an equation satisfied by the polar coordinates of the point P and use it to sketch the Cissoid of Diocles.



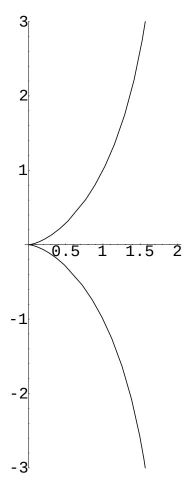
(15 points)

. . . .

The polar equation of the line $x = 2$ is $r = 2/\cos\theta$. The point A lies on the circle with polar equation $r = 2\cos\theta$. Consequently, the polar equation of curve traced out by the point P is

$$r = \frac{2}{\cos\theta} - 2\cos\theta.$$

Using a calculator,we see that the required graph is



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6. Find the eccentricity of the conic section with polar equation

$$r = \frac{7}{2 - 5\sin\theta}.$$

Identify the conic, give the equation of the directrix and sketch the conic.

(15 points)

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The equation can be rewritten as

$$r = \frac{7/2}{1 + (5/2)\cos(\theta + \pi/2)}.$$

Consequently, the eccentricity is 5/2 and so the conic section is a hyperbola. It can be obtained by rotating the hyperbola

$$r = \frac{7/2}{1 + (5/2)\cos\theta}$$

through $-\pi/2$ about the origin. The latter hyperbola has directrix $x = 7/5$. Consequently our hyperbola has directrix $y = -7/5$. Its graph is