# CALCULUS III FALL 1999

# REVIEW 2

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**0.** Great fleas have little fleas upon their backs to bite 'em, and little fleas have lesser fleas, and so ad infinitum.

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**1.** Find the following limits if they exist:

$$\lim_{n\to\infty} \frac{2n^2-1}{3n^2+15n-2}, \qquad \lim_{n\to\infty} (\sqrt{n+1}-\sqrt{n-1}), \qquad \lim_{n\to\infty} \frac{n!}{n^n},$$

$$\lim_{n\to\infty} (\sqrt{n^2+n}-\sqrt{n^2+1}).$$

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**2.** Do the same thing for the following:

$$\lim_{n\to\infty} \frac{\ln(2n)}{\ln(3n)} \qquad \lim_{n\to\infty} \sqrt[n]{n^2+n} \qquad \lim_{n\to\infty} \sqrt[n]{a^n+b^n} \qquad \lim_{n\to\infty} \frac{\alpha(n)}{n},$$

where $\alpha(n)$ is the number of primes which divide $n$.

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**3.** Give an example of a sequence that is:

(a) convergent but not monotonic
(b) bounded but not monotonic
(c) monotone but not convergent
(d) monotone decreasing and unbounded
(e) monotone decreasing and convergent
(f) unbounded but not monotone

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**4.** Show that the following sequences converge and find their limits:

$$a_1 = \sqrt{3}, \quad a_{n+1} = \sqrt{a_n + 2}; \qquad\qquad a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}$$

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**5.** Identify the function $f(x) = \lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi x)^{2k})$ - this is a little tricky, but lots of fun.

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**6.** Determine which of the following series are convergent:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2-1}} \qquad \sum_{n=1}^{\infty} \frac{n^2}{n!} \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n} \qquad \sum_{n=1}^{\infty} \frac{n^2-1}{n^4-1} \qquad \sum_{n=1}^{\infty} \sin\frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{5^n n!}{(2n)^n}$$

In the case of one of the convergent series above, estimate the sum accurately to 3 decimal places.

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**7.** For which values of $x$ do the following series converge?

$$\sum_{n=1}^{\infty} (\sin x)^n \qquad\qquad \sum_{n=1}^{\infty} (1+x)^n \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{(1-x)^n}.$$

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**8.** Determine which of the following series converge:

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n\sqrt{n}} \qquad \sum_{n=1}^{\infty} (-1)^{n+1}\frac{n-1}{n} \qquad \sum_{n=1}^{\infty} n\sin\frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$$

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**9.** Find the interval of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n^3+1} \qquad\qquad \sum_{n=0}^{\infty} \frac{(-2)^{n+1}x^n}{2n+3} \qquad\qquad \sum_{n=0}^{\infty} \frac{3^n x^{3n}}{(3n)!}.$$

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# REVIEW 2 – ANSWERS

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**1.**

$$\frac{2n^2-1}{3n^2+15n-2} \to \frac{2}{3}$$

$$\sqrt{n+1}-\sqrt{n-1} = \frac{1}{\sqrt{n+1}+\sqrt{n-1}} \to 0$$

$$\frac{n!}{n^n} = \frac{1.2\ldots n}{n\ldots n} \leqslant \frac{1}{n} \to 0$$

$$\sqrt{n^2+n}-\sqrt{n^2+1} = \frac{n-1}{\sqrt{n^2+n}+\sqrt{n^2+1}} \to \frac{1}{2}.$$

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**2.**

$$\frac{\ln(2n)}{\ln(3n)} = \frac{\ln 2+\ln n}{\ln 3+\ln n} \to 1$$

$$\sqrt[n]{n^2+n} = n^{1/n}(n+1)^{1/n} = n^{1/n}\left(\frac{n+1}{n}\right)^{1/n} n^{1/n} \to e.$$

If $a \leqslant b$ then

$$\sqrt[n]{a^n+b^n} = b\left(1+\left(\frac{a}{b}\right)^n\right)^{1/n} \to b, \qquad \text{in general } \sqrt[n]{a^n+b^n} \to \max\{a,b\}.$$

Note that if $\alpha(n)=m$ then there are $m$ primes $p_1,\ldots,p_m$ which all divide $n$. Hence $p_1\ldots p_m \leqslant n$, on the other hand $p_1\ldots p_m \geqslant 2^m$. Hence $m\ln 2 \leqslant \ln n$. Thus

$$\frac{\alpha(n)}{n} \leqslant \frac{\ln n}{n\ln 2} \to 0,$$

by L'Hôpital's rule.

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**3.** a) $(-1)^n/n$ b) $(-1)^n$ c) $n$ d) $-n$ e) $1/n$ f) $(-1)^n n$.

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**4.** $a_{n+1}^2 - a_n^2 = -(a_n - 2)(a_n + 1)$ and $a_{n+1}^2 - 4 = a_n - 2$. So $0 < a_n < 2$ for all $n$ since $a_1 < 2$. Furthermore $a_{n+1} > a_n$ for all $n$ and so we have a bounded monotonic sequence. It follows that the sequence is convergent and the limit $\ell$ satisfies $\ell^2 = \ell + 2$. This implies $\ell = -1$ or 2. Clearly the limit is 2.
$a_{n+1}^2 - a_n^2 = a_n(2 - a_n)$ and $a_{n+1}^2 - 4 = 2(a_n - 2)$. So $0 < a_n < 2$ for all $n$ since $a_1 < 2$. Furthermore $a_{n+1} > a_n$ for all $n$ and so we have a bounded monotonic sequence. It follows that the sequence is convergent and the limit $\ell$ satisfies $\ell^2 = 2\ell$. This implies $\ell = 0$ or 2. Clearly the limit is 2.

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**5.** If $x$ is rational, then $n!\pi x$ is an integer multiple of $\pi$ for all sufficiently large values of $n$. Hence $(\cos n!\pi x)^{2k} = 1$ for all $k$ and all sufficiently large $n$. Hence $f(x) = 1$ if $x$ is rational. If $x$ is irrational, then $|\cos n!\pi x|^{2k} < 1$ for each $n$. Hence $f(x) = 0$ if $x$ is irrational.

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**6.** Comparison with $\sum 1/n^{2/3}$ shows that the series diverges.

$$\frac{(n+)^2 n!}{(n+1)!n^2} = \frac{n+1}{n^2} \to 0,$$

so the ratio test shows we have convergence.

$$\frac{\ln n}{n} > \frac{1}{n}, \qquad \text{for sufficiently large } n$$

so the comparison test shows that the series diverges.

$$\frac{n^2-1}{n^4-1} = \frac{1}{n^2+1},$$

so comparison with $\sum 1/n^2$ shows that this converges.

$$\frac{1/n}{\sin 1/n} \to 1,$$

so the comparison test shows that this series diverges.

$$\frac{5^{n+1}(n+1)!(2n)^n}{(2n+2)^{n+1}5^n n!} = \frac{5n^n}{2(n+1)^n} \to \frac{5}{2e} < 1,$$

So the ratio test shows that the series converges.

We'll estimate $\sum_{n=2}^{\infty}(n^2-1)/(n^4-1) = \sum_{n=2}^{\infty} 1/(n^2+1)$. We can apply the integral test to this to see that

$$R_n \leqslant \int_n^{\infty} \frac{dx}{1+x^2} = \lim_{A\to\infty}\Big[\tan^{-1} x\Big]_n^A = \frac{\pi}{2} - \tan^{-1} n = \tan^{-1}\frac{1}{n}.$$

So we choose $n$ with $\tan^{-1}(1/n) < 0.0005$. We know that $\tan^{-1} x < x$ so it suffices to choose $n = 2,000$. The required estimate is $s_{2,000} = 0.576$.

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**7.** For convergence, we require $|\sin x| < 1$, which means $x$ is not an odd multiple of $\pi$. For the second series, we require $|1 + x| < 1$ which means $x$ is in the interval $(-2, 0)$. Finally, we require $|1 - x| > 1$ for convergence. This means either $x < 0$ or $x > 2$.

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**8.** The corresponding absolute series is $\sum_{n=1}^{\infty} 1/n^{3/2}$ which converges. So we have absolute convergence.
This series diverges since its terms do not approach zero.
Note that $n \sin(1/n) \to 1$ and so this series diverges too.
This series does not converge absolutely, by comparison with $\sum_{n=1}^{\infty} 1/\sqrt{n}$. On the other hand, the alternating series test can be applied to show the series converges conditionally.

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**9.** We use the ratio test in all three cases. For the first series, we have

$$\frac{|x|^{n+1}(n^3+1)}{((n+1)^3+1)|x|^n} = \frac{|x|(n^3+1)}{(n+1)^3+1} \to |x|.$$

When $x = \pm 1$, the corresponding series of absolute values is $\sum_{n=1}^{\infty} 1/(n^3 + 1)$ which converges. So the interval of convergence is $[-1, 1]$.

For the second series,

$$\frac{2^{n+2}|x|^{n+1}(2n+3)}{(2n+5)2^{n+1}|x|^n} = \frac{2|x|(2n+3)}{2n+5} \to 2|x|.$$

When $x = 1/2$, the series is $-2\sum_{n=1}^{\infty}(-1)^n/(2n+3)$ which converges (alternating series test). When $x = -1/2$ the series is $-2\sum_{n=1}^{\infty} 1/(2n+3)$ which diverges. So the interval of convergence is $(-1/2, 1/2]$.

For the third series

$$\frac{3^{n+1}|x|^{3n+3}(3n)!}{(3n+3)!3^n|x|^{3n}} = \frac{3|x|^3}{(3n+3)(3n+2)(3n+1)} \to 0.$$

So the series converges for all $x$.