Paramodular forms in CAP representations of GSp(4)

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Abstract. We explicitly determine the non-tempered local Arthur packets for GSp(4) of Howe - Piatetski-Shapiro type, Saito-Kurokawa type and Soudry type. As a consequence we show that Gritsenko lifts are the only paramodular forms that can occur in global CAP representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$.

Contents

Introduction

Arthur [3] has classified the discrete automorphic spectrum of symplectic and split orthogonal groups. For the group $\mathrm{PGSp}(4) \cong \mathrm{SO}(5)$, the discretely appearing automorphic representations come in finite or infinite packets, of which there are six types. The "general" type **(G)** consists of those representations that lift to cusp forms on GL(4). The Yoshida type **(Y)** can be characterized as representations whose L-functions are of the form $L(s, \pi_1)L(s, \pi_2)$ with distinct cuspidal, automorphic representations on GL(2). At least conjecturally **(G)** and **(Y)** consist of everywhere tempered representations. Then there are three non-tempered types **(Q)**, **(P)** and **(B)**, associated with the three conjugacy classes of parabolic subgroups. These consist essentially of CAP representations with respect to the Klingen parabolic Q, the Siegel parabolic P, and the Borel subgroup B, respectively. Finally, there is the type **(F)** consisting of one-dimensional representations. See [2], [23] for a more detailed description of the six types.

For a positive integer N, let $K(N)$ be the paramodular group of level N. Holomorphic Siegel modular forms of degree 2 with respect to $K(N)$ are known as ***paramodular forms***. These are well behaved in many ways; for example, there is a theory of old- and newforms [17], and cuspidal newforms admit a strong multiplicity one theorem [23]. Paramodular forms of weight 2 are also the ones appearing in the ***paramodular conjecture*** formulated in [4].

"Most" paramodular forms appear in packets of type **(G)**. As observed in Lemma 2.5 of [23], packets of type **(Y)** cannot contain any paramodular forms. It is known that ***Gritsenko*** or

Saito-Kurokawa liftings appear in packets of type **(P)**. In this note we will prove that packets of type **(P)** do not contain any paramodular forms besides these liftings, and that packets of type **(Q)** or **(B)** do not contain any paramodular forms at all. As an application we can slightly strengthen the paramodular strong multiplicity one theorem of [23].

Our method is to calculate the local Arthur packets for types **(Q)**, **(P)** and **(B)** explicitly. Once this is done, one can use the local theory of [18] to look up which representations contain paramodular vectors. It turns out that for types **(Q)** and **(B)**, and working over the number field $\mathbb{Q}$, there always exists a non-archimedean place for which none of the elements in the packet is paramodular.

Local Arthur packets for GSp(4) have one or two elements, the size being determined by a centralizer group. Each local packet contains a "base point", which is easy to determine by general principles. The main difficulty is to determine the "non base point" in the cases where the packet has two elements. For types **(P)** and **(B)**, we do this in an indirect way, using the fact from [15] (see also [5] and [20]) that the global representations in question can be constructed as theta liftings from the metaplectic group $\widetilde{\mathrm{SL}}(2)$. Hence the representations in a local packet can be determined by calculating certain cases of the local theta correspondence. For type **(Q)**, it turns out that the 2-element packets coincide with those of type **(B)**, so no additional work is necessary. Tables 1, 2 and 3 summarize the local packets in the three cases.

Some of the results in this note are not new. For example, the local and global packets of type **(Q)** have also been determined in [8]. Some explicit information for type **(P)** is already contained in [20] and [5]. Still we found it useful to summarize the constructions in all cases and present the local packets using standard notation for GSp(4). We also give information on K-types in the real case, which is useful for applications to Siegel modular forms.

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Notation. For most of this note we work with the group $\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) \,|\, {}^tgJg = \mu(g)J\}$ defined by the symplectic form

$$J = \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix},$$

except for Sect. 5, where we will switch to the "classical" symplectic form $\left[\begin{smallmatrix} 0 & 1_2 \\ -1_2 & 0 \end{smallmatrix}\right]$. The group Sp(4) is the subgroup consisting of elements for which the scalar $\mu(g)$ is 1. If F is a local field, then let L_F be its Weil group if F is archimedean, and its Weil-Deligne group if F is non-archimedean. If F is non-archimedean, we use the classification of irreducible, admissible representations of $\mathrm{GSp}(4,F)$ into types I, IIa, IIb, ..., as explained in Sect. 2.2 of [18].

1 Global packets of type (B), (P) and (Q)

Let F be an algebraic number field, and $\mathbb{A}$ its ring of adeles. Global Arthur parameters are formal objects of the form $\sum_i \mu_i \boxtimes \nu(n_i)$, where μ_i is a self-dual, unitary, cuspidal automorphic representation of $\mathrm{GL}(m_i,\mathbb{A})$, and $\nu(n)$ is the irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ of dimension n. In the case of GSp(4), these parameters come in six different types. In this work we are interested in the parameters which in [23] were called of type **(B)**, **(P)** and **(Q)**. Their description is as follows.

(B) $\psi = (\chi_1 \boxtimes \nu(2)) \boxplus (\chi_2 \boxtimes \nu(2))$, where χ_1, χ_2 are distinct quadratic Hecke characters. These are the parameters of Howe - Piatetski-Shapiro type.

(P) $\psi = (\mu \boxtimes 1) \boxplus (\sigma \boxtimes \nu(2))$, where μ is a unitary, cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character, and σ is a quadratic Hecke character. These are the parameters of Saito - Kurokawa type.

(Q) $\psi = \mu \boxtimes \nu(2)$, where μ is a self-dual, unitary, cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with non-trivial central character. These are the parameters of Soudry type. The central character ξ of μ determines a quadratic extension E of F. There exists a character θ of $\mathbb{A}_E^\times$ such that $\mu = \mathcal{AI}_{E/F}(\theta)$, i.e., μ is obtained from θ by automorphic induction.

Given such a global parameter ψ and a place v of F, there is an associated local Arthur parameter ψ_v. These are maps

$$\psi_v : L_{F_v} \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{Sp}(4, \mathbb{C}). \tag{1}$$

By (1.4) - (1.7) of [23], their explicit form is as follows, where $w \in L_{F_v}$.

(B) We factor $\chi_i = \otimes \chi_{i,v}$ and identify $\chi_{i,v}$ with a character of L_{F_v}. Then ψ_v is given by

$$(w,1) \longmapsto \begin{bmatrix} \chi_{1,v}(w) &&& \\ & \chi_{2,v}(w) && \\ && \chi_{2,v}(w) & \\ &&& \chi_{1,v}(w) \end{bmatrix}, \qquad (1, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) \longmapsto \begin{bmatrix} a &&& b \\ & a & b & \\ & c & d & \\ c &&& d \end{bmatrix}. \tag{2}$$

(P) We factor $\mu = \otimes \mu_v$ and $\sigma = \otimes \sigma_v$. Let

$$\phi_v : L_F \to \mathrm{SL}(2, \mathbb{C}), \qquad \phi(w) = \left[\begin{smallmatrix} \phi_{v,1}(w) & \phi_{v,2}(w) \\ \phi_{v,3}(w) & \phi_{v,4}(w) \end{smallmatrix}\right], \tag{3}$$

be the L-parameter of the irreducible, admissible representation μ_v of $\mathrm{PGL}(2, F_v)$. Then ψ_v is given by

$$(w,1) \longmapsto \begin{bmatrix} \sigma_v(w) &&& \\ & \phi_{v,1}(w) & \phi_{v,2}(w) & \\ & \phi_{v,3}(w) & \phi_{v,4}(w) & \\ &&& \sigma_v(w) \end{bmatrix}, \qquad (1, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) \longmapsto \begin{bmatrix} a &&& b \\ & 1 && \\ && 1 & \\ c &&& d \end{bmatrix}. \tag{4}$$

(Q) We factor $\mu = \otimes \mu_v$. The L-parameter $\phi_v : L_{F_v} \to \mathrm{GL}(2, \mathbb{C})$ of μ_v can be arranged in such a way that it takes values in the group

$$\mathrm{O}(2, \mathbb{C}) = \{A \in \mathrm{GL}(2, \mathbb{C}) \mid {}^tA \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right] A = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]\}. \tag{5}$$

Then ψ_v is given by

$$(w,1) \longmapsto \begin{bmatrix} \phi_v(w) & \\ & \phi_v(w) \end{bmatrix}, \qquad (1, \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) \longmapsto \begin{bmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{bmatrix}. \tag{6}$$

To ψ_v is attached a finite set Π_{ψ_v} of irreducible, admissible representations of $\mathrm{PGSp}(4, F_v)$ according to Theorem 1.5.1 of [3]. Let Z be the center of $\mathrm{Sp}(4,\mathbb{C})$. The elements of the local Arthur packet Π_{ψ_v} are fibered over the characters of the centralizer group

$$\mathcal{S}_{\psi_v} = S_{\psi_v}/S^0_{\psi_v}Z, \tag{7}$$

where S_{ψ_v} is the centralizer of the image of ψ_v, and $S^0_{\psi_v}$ is its identity component. It is easy to see that the groups $\mathcal{S}_{\psi_v}$ are as follows.

- **(B)** $\mathcal{S}_{\psi_v}$ has two elements in all cases.
- **(P)** $\mathcal{S}_{\psi_v}$ is trivial if μ_v is a principal series representation, and otherwise has two elements.
- **(Q)** We factor $\theta = \otimes\theta_w$. If v does not split in E, and w is the unique place of E lying above v, then the L-parameter of μ_v equals $\mathrm{ind}_{W_{E_w}}^{W_{F_v}}(\theta_w)$. If v splits in E, and w_1, w_2 are the two places of E lying above v, then $\theta_w := (\theta_{w_1}, \theta_{w_2})$ is a pair of characters of $F_v^\times$, and μ_v is the principal series representation $\theta_{w_1} \times \theta_{w_2}$; since the central character ξ_v is trivial, we have in fact $\theta_{w_1}\theta_{w_2} = 1$. In either case we write $\mu_v = \mathcal{AI}_{E_w/F_v}(\theta_w)$. Then $\mathcal{S}_{\psi_v}$ is trivial if θ_w is not Galois-invariant (which in the split case means that $\theta_{w_1} \neq \theta_{w_2}$), and otherwise has two elements.

Since $\mathcal{S}_{\psi_v}$ has at most two elements, we may think of the fibration over the characters of $\mathcal{S}_{\psi_v}$ as a map $\epsilon : \Pi_{\psi_v} \to \{\pm 1\}$. If $\mathcal{S}_{\psi_v}$ is trivial, then ϵ is $+1$ for all representations in the local packet. It will turn out later that the map ϵ is in fact injective. Since this is not true in general (see the comments after Theorem 1.5.1 of [3]), we will however not use this fact.

Our goal is to determine the local packets Π_{ψ_v} explicitly. By Proposition 7.4.1 of [3], Π_{ψ_v} contains the irreducible representations with L-parameter

$$\phi_{\psi_v}(w) = \psi(w, \begin{bmatrix} |w|_v^{1/2} & \\ & |w|_v^{-1/2} \end{bmatrix}), \qquad w \in L_{F_v}. \tag{8}$$

In each case this turns out to be a single unitary representation, which can be viewed as a "base point" π_v^+ in the local packet. For non-archimedean v, using (2), (4) and (6), π_v^+ can be read off Table A.7 of [18].

- **(B)** π_v^+ is the Langlands quotient of the Borel induced representation

$$\chi_{1,v}\chi_{2,v}|\cdot|_v \times \chi_{1,v}\chi_{2,v} \rtimes \chi_{2,v}|\cdot|_v^{-1/2}. \tag{9}$$

  If $\chi_{1,v} \neq \chi_{2,v}$, it is the representation $L(\chi_{1,v}\chi_{2,v}|\cdot|_v, \chi_{1,v}\chi_{2,v} \rtimes |\cdot|_v^{-1/2}\chi_{2,v})$ of type Vd, and if $\chi_{1,v} = \chi_{2,v}$ it is the representation $L(|\cdot|_v, 1_{F_v^\times} \rtimes |\cdot|_v^{-1/2}\chi_{1,v})$ of type VId.

- **(P)** π_v^+ is the Langlands quotient of the Siegel induced representation

$$|\cdot|_v^{1/2}\sigma_v\mu_v \rtimes |\cdot|_v^{-1/2}\sigma_v. \tag{10}$$

  There are four possibilities, depending on μ_v and σ_v:

  - If μ_v is a principal series representation $\chi_v \times \chi_v^{-1}$ with a character χ_v of $F_v^\times$, then π_v^+ is the representation $\chi_v\sigma_v 1_{\mathrm{GL}(2)} \rtimes \chi_v^{-1}$ of type IIb.

- If $\mu_v = \chi_v \mathrm{St}_{\mathrm{GL}(2)}$, where χ_v is a quadratic character different from σ_v, then π_v^+ is the representation $L(|\cdot|_v^{1/2}\chi_v\sigma_v\mathrm{St}_{\mathrm{GL}(2)}, |\cdot|_v^{-1/2}\sigma_v)$ of type Vb.
- If $\mu_v = \sigma_v \mathrm{St}_{\mathrm{GL}(2)}$, then π_v^+ is the representation $L(|\cdot|_v^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, |\cdot|_v^{-1/2}\sigma_v)$ of type VIc.
- If μ_v is supercuspidal, then π_v^+ is the representation $L(|\cdot|_v^{1/2}\sigma_v\mu_v, |\cdot|_v^{-1/2}\sigma_v)$ of type XIb.

(Q) π_v^+ is the Langlands quotient of the Klingen induced representation

$$|\cdot|_v\xi_v \rtimes |\cdot|_v^{-1/2}\mu_v. \tag{11}$$

There are four possibilities, depending on whether ξ_v is trivial or not, and whether θ_w (defined as above) is Galois-invariant or not:

- If $\xi_v \neq 1$ and θ_w is not Galois-invariant (i.e., μ_v is supercuspidal), then π_v^+ is the representation $L(|\cdot|_v\xi_v, |\cdot|_v^{-1/2}\mu_v)$ of type IXb.
- If $\xi_v \neq 1$ and $\theta_w = \sigma_v \circ N_{E_w/F_v}$ with a quadratic character σ_v of $F_v^\times$ (i.e., $\mu_v = \sigma_v \times (\xi_v\sigma_v)$), then π_v^+ is the representation $L(|\cdot|_v\xi_v, \xi_v \rtimes |\cdot|_v^{-1/2}\sigma_v)$ of type Vd.
- If $\xi_v = 1$ and $\theta_w = (\theta_{w_1}, \theta_{w_2})$ with $\theta_{w_1} \neq \theta_{w_2}$, then π_v^+ is the representation $\theta_{w_1}\theta_{w_2}^{-1} \rtimes \theta_{w_2}1_{\mathrm{GSp}(2)}$ of type IIIb.
- If $\xi_v = 1$ and $\theta_w = (\sigma_v, \sigma_v)$ with a quadratic character σ_v of $F_v^\times$ (i.e., $\mu_v = \sigma_v \times \sigma_v$), then π_v^+ is the representation $L(|\cdot|_v, 1_{F_v^\times} \rtimes |\cdot|_v^{-1/2}\sigma_v)$ of type VId.

It follows from Tables A.1 and A.2 of [18] that π_v^+ is unitary, non-tempered and non-generic in all cases. To π_v^+ is attached the sign $\epsilon(\pi_v^+) = 1$.

The global Arthur packet is defined as

$$\Pi_\psi := \big\{\pi = \otimes\pi_v \mid \pi_v \in \Pi_{\psi_v},\ \pi_v = \pi_v^+ \text{ for almost all } v\big\}. \tag{12}$$

The "global base point" in each packet is obtained by taking π_v^+ at each place. Hence the global base point is the isobaric constituent of the following globally induced representation:

(B) $\chi_1\chi_2|\cdot| \times \chi_1\chi_2 \rtimes \chi_2|\cdot|^{-1/2}$.

(P) $|\cdot|^{1/2}\sigma\mu \rtimes |\cdot|^{-1/2}\sigma$.

(Q) $|\cdot|\xi \rtimes |\cdot|^{-1/2}\mu$.

Every element of Π_ψ is near equivalent to the global base point. Hence, cuspidal elements of Π_ψ are CAP with respect to the Borel subgroup B, the Siegel parabolic P, or the Klingen parabolic Q, respectively.

1.1 Lemma. *Let ψ be any Arthur parameter for* $\mathrm{GSp}(4)$. *The discrete automorphic elements of* Π_ψ *comprise a near-equivalence class of all representations in the discrete automorphic spectrum of* $\mathrm{PGSp}(4, \mathbb{A})$.

Proof. It is clear from the definition (12) that the elements of Π_ψ are near-equivalent. Assume that π occurs in the discrete, automorphic spectrum of $\mathrm{GSp}(4,\mathbb{A})$ and is near equivalent to the elements in Π_ψ. We have to show that $\pi \in \Pi_\psi$. Assume first that ψ is of type **(G)**. Then $\psi = \mu \boxtimes 1$ with a self-dual, symplectic, unitary, cuspidal automorphic representation μ of $\mathrm{GL}(4,\mathbb{A})$. By the definitions involved, the partial L-functions of the elements of Π_ψ are equal to $L^S(s,\mu)$, where S is a finite set of places. Hence $L^S(s,\pi) = L^S(s,\mu)$ for large enough S. By considering the partial L-functions of the various types of Arthur packets, as in Table 1 of [23], and keeping in mind the classification of automorphic representations of $\mathrm{GL}(n)$, as in Theorem (4.4) of [10], we see that π must be of type **(G)**. The strong multiplicity one theorem for $\mathrm{GL}(4)$ then implies that $\pi \in \Pi_\psi$.

Similar arguments apply for ψ of one of the other types. ∎

We return to ψ of type **(B)**, **(P)** or **(Q)**. If we twist all elements of a packet Π_ψ by a fixed quadratic Hecke character χ, then we obtain another packet of the same type. More precisely,

$$\chi \otimes \Pi_\psi = \Pi_{\chi\otimes\psi}, \tag{13}$$

where

$$\chi \otimes \psi = \begin{cases} (\chi\chi_1 \boxtimes \nu(2)) \boxplus (\chi\chi_2 \boxtimes \nu(2)), \\ (\chi\mu \boxtimes 1) \boxplus (\chi\sigma \boxtimes \nu(2)), \\ \chi\mu \boxtimes \nu(2), \end{cases} \tag{14}$$

for

$$\psi = \begin{cases} (\chi_1 \boxtimes \nu(2)) \boxplus (\chi_2 \boxtimes \nu(2)) & \text{of type } \textbf{(B)}, \\ (\mu \boxtimes 1) \boxplus (\sigma \boxtimes \nu(2)) & \text{of type } \textbf{(P)}, \\ \mu \boxtimes \nu(2) & \text{of type } \textbf{(Q)}. \end{cases} \tag{15}$$

Let $\pi = \otimes \pi_v$ be an element of Π_ψ. The multiplicity $m(\pi)$ with which π appears in the discrete automorphic spectrum is either 0 or 1. Arthur's multiplicity formula characterizes those π's with $m(\pi) = 1$, as follows. Let $\epsilon(\pi) = \prod_v \epsilon(\pi_v)$.

- **(B)** $m(\pi) = 1$ if and only if $\epsilon(\pi) = 1$.
- **(P)** $m(\pi) = 1$ if and only if $\epsilon(\pi) = \varepsilon(1/2, \sigma \otimes \mu)$.
- **(Q)** $m(\pi) = 1$ for all $\pi \in \Pi_\psi$. Since there is no sign condition, we say that these packets are *stable.*

Hence, for types **(B)** and **(Q)**, the global base point appears in the discrete automorphic spectrum, and for type **(P)** it does so if and only if $\varepsilon(1/2, \sigma \otimes \mu) = 1$.

1.2 Lemma. *Let ψ be an Arthur parameter of type* **(B)**, **(P)** *or* **(Q)**. *Let π be an element of Π_ψ that appears in the discrete automorphic spectrum. Then π is cuspidal if and only if one of the following conditions is satisfied:*

i) π is not the global base point.

ii) $\psi = (\mu \boxtimes 1) \boxplus (\sigma \boxtimes \nu(2))$ is of type **(P)**, *π is the global base point (hence assuming $\varepsilon(1/2, \sigma \otimes \mu) = 1$), and $L(1/2, \sigma \otimes \mu) = 0$.*

Proof. The residual spectrum of $\mathrm{GSp}(4,\mathbb{A})$ is explicitly described in Sect. 7 of [11]. From this description it is easy to see that if $\pi \in \Pi_\psi$ is in the residual spectrum, then π must be the global base point, and if ψ is of type **(P)**, then in addition $L(1/2, \sigma \otimes \mu)$ must be non-zero. ∎

2 Local packets for type (B)

To determine the local packets for Arthur parameters of Howe - Piatetski-Shapiro type, we first recall some facts about the theta correspondence between the metaplectic group $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ and the group $\mathrm{SO}(5,\mathbb{A}) \cong \mathrm{PGSp}(4,\mathbb{A})$; see [15], [26], [5]. The structure of the Weil representations of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ is well known. Locally, they are parametrized by quadratic characters χ_v of $F_v^\times$. Given such a χ_v, the local Weil representation $\tilde\pi_{\chi_v}$ splits into two irreducible parts, the *even* Weil representation $\pi^+_{\chi_v}$ and the supercuspidal *odd* Weil representation $\pi^-_{\chi_v}$. Globally, let $\chi = \otimes\chi_v$ be a non-trivial quadratic Hecke character, and let S be a finite set of places of even cardinality. Then

$$\tilde\pi_\chi^S := \Big(\bigotimes_{v\in S} \tilde\pi^-_{\chi_v}\Big) \otimes \Big({\bigotimes_{v\notin S}}' \,\tilde\pi^+_{\chi_v}\Big) \tag{16}$$

defines a representation of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$. These are the irreducible, automorphic constituents of the global Weil representation. Evidently, their near-equivalence classes are obtained by fixing χ and varying S. We note that, in order for a collection of local metaplectic representations to define a representation of the global $\widetilde{\mathrm{SL}}(2,\mathbb{A})$, the parity condition (1) on page 280 of [26] has to be satisfied. For the Weil representations on the right side of (16) this parity condition is equivalent to the cardinality of S being even.

We consider the non-archimedean local theta liftings, temporarily omitting the subindex v. Thus let F be a non-archimedean local field of characteristic zero, and let χ by a quadratic character of $F^\times$. The theta liftings of the even and odd Weil representations are as follows.

$\widetilde{\mathrm{SL}}(2,F)$	$\mathrm{GSp}(4,F)$	type	L-parameter
$\tilde\pi^+_\chi$	$L(\nu\chi, \chi \rtimes \nu^{-1/2})$	Vd	$\chi\varphi_1 \oplus \varphi_1$
$\tilde\pi^-_\chi$	$\delta^*([\chi,\nu\chi], \nu^{-1/2})$	Va*	$\chi\varphi_{\mathrm{St}} \oplus \varphi_{\mathrm{St}}$
$\tilde\pi^+_1$	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2})$	VId	$\varphi_1 \oplus \varphi_1$
$\tilde\pi^-_1$	$L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2})$	VIc	$\varphi_1 \oplus \varphi_{\mathrm{St}}$

(17)

In this table, φ_1 denotes the L-parameter of the trivial representation of $\mathrm{GL}(2,F)$, given in (ρ, N) form by

$$\varphi_1 : \; w \longmapsto \begin{bmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{bmatrix}, \qquad N = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \tag{18}$$

and φ_{St} denotes the L-parameter of the Steinberg representation of $\mathrm{GL}(2,F)$, given in (ρ, N) form by

$$\varphi_{\mathrm{St}} : \; w \longmapsto \begin{bmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{bmatrix}, \qquad N = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]. \tag{19}$$

The last column in (17) lists the L-parameter of the $\mathrm{GSp}(4,F)$ representation as a 4-dimensional representation of L_F; see Table A.7 of [18].

We indicate how to find the liftings in (17). Since $\mathrm{PGSp}(4,F) \cong \mathrm{SO}(5,F)$, the non-supercuspidal cases Vd, VId and VIc follow from the well known theta correspondence between $\widetilde{\mathrm{SL}}(2,F)$ and $\mathrm{SO}(3,F) \cong \mathrm{PGL}(2,F)$, together with a tower argument. For Va*, one can use Proposition 5.8 of [20] (with $\tau = \chi 1_{D^\times}$), which implies that the theta lifting of $\tilde{\pi}_\chi^-$ coincides with the theta lifting of a one-dimensional representation of a certain group $\mathrm{GO}(4)$. The latter lifting has been calculated in Theorem 4.6.3 of [19].

We note that the definition of the representations $\tilde{\pi}_\chi^\pm$ depends on the choice of an additive character of F. The definition of the theta correspondence also depends on such a choice. We choose these two characters to be the same, in which case the $\mathrm{GSp}(4,F)$ representations in (17) are independent of which character is chosen.

We return to F being global.

2.1 Lemma. *Let χ be a nontrivial quadratic character of $F^\times\backslash\mathbb{A}^\times$. Consider the Arthur parameter $\psi = (\chi \boxtimes \nu(2)) \boxplus (1 \boxtimes \nu(2))$ of Howe - Piatetski-Shapiro type. As S runs through the finite sets of places of F with even cardinality, the theta liftings of the Weil representations $\tilde{\pi}_\chi^S$ to $\mathrm{SO}(5,\mathbb{A}) \cong \mathrm{PGSp}(4,\mathbb{A})$ run through the discrete automorphic elements of the global packet Π_ψ.*

Proof. Consider first $\tilde{\pi}_\chi^S$ for non-empty S. It is a cuspidal representation of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$, because $\tilde{\pi}_{\chi_v}^-$ is supercuspidal. Its first occurrence in the tower $\mathrm{SO}(2n+1,\mathbb{A})$ must be cuspidal. This first occurrence cannot happen with $\mathrm{SO}(3,\mathbb{A}) \cong \mathrm{PGL}(2,\mathbb{A})$, since the local lifting of $\tilde{\pi}_{\chi_v}^+$ is one-dimensional. Hence, by stable range, $\tilde{\pi}_\chi^S$ lifts to a cusp form on $\mathrm{SO}(5,\mathbb{A}) \cong \mathrm{PGSp}(4,\mathbb{A})$. By (17), this cusp form is CAP to the globally induced $\chi|\cdot| \times \chi \rtimes |\cdot|^{-1/2}$, and therefore an element of the packet Π_ψ. For empty S, the lift of $\tilde{\pi}_\chi^S$ is the isobaric constituent of $\chi|\cdot| \times \chi \rtimes |\cdot|^{-1/2}$, hence the base point in Π_ψ.

Conversely, let π be an element of Π_ψ which is not the base point. Then π is cuspidal by Lemma 1.2. As we saw in the previous section, π is CAP to $\chi|\cdot| \times \chi \rtimes |\cdot|^{-1/2}$. By Theorems 2.2 and 2.4 of [15], π is a theta lifting of a Weil representation of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$. More precisely, by comparing local components almost everywhere, we see that π must be a lifting of $\tilde{\pi}_\chi^S$ for some S. ∎

It follows from Lemma 2.1 that the local Arthur packets Π_{ψ_v} for $\psi = (\chi \boxtimes \nu(2)) \boxplus (1 \boxtimes \nu(2))$ contain two elements, namely the theta liftings of $\tilde{\pi}_{\chi_v}^\pm$. For arbitrary ψ of type **(B)**, the local packets are then obtained by twisting; see (13), (14), (15). We thus obtain from (17) the non-archimedean local packets summarized in Table 1. To determine the local signs given in the last column of Table 1, we can argue as follows. Since there is a parity condition on S in Lemma 2.1, the global packet Π_ψ is unstable, meaning a discrete automorphic element $\pi = \otimes\pi_v$ of Π_ψ will no longer be automorphic if π_v is replaced by its partner in the local packet for a single place v. The condition $\epsilon(\pi) = 1$ in Arthur's multiplicity formula hence implies that the two representations in a local packet must be assigned different signs. Since the base point in each local packet always has the sign $+1$, the non-base point must have sign -1.

Table 1: Local Arthur packets Π_ψ of Howe - Piatetski-Shapiro type (Borel packets, type **(B)**). The local Arthur parameter ψ is determined by a pair (χ_1, χ_2) of quadratic characters of $F^\times$; see (2). For the archimedean K-type H, see (20); for A, C and D, see (21).

(χ_1, χ_2)	$\mathrm{GSp}(4, F)$	type	L-parameter	ϵ
	non-archimedean case			
$\chi_1 \neq \chi_2$	$L(\chi_1\chi_2\nu, \chi_1\chi_2 \rtimes \nu^{-1/2}\chi_2)$	Vd	$\chi_1\varphi_1 \oplus \chi_2\varphi_1$	$+1$
	$\delta^*([\chi_1\chi_2, \nu\chi_1\chi_2], \nu^{-1/2}\chi_2)$	Va*	$\chi_1\varphi_{\mathrm{St}} \oplus \chi_2\varphi_{\mathrm{St}}$	-1
$\chi_1 = \chi_2$	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\chi_1)$	VId	$\chi_1\varphi_1 \oplus \chi_1\varphi_1$	$+1$
	$L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\chi_1)$	VIc	$\chi_1\varphi_1 \oplus \chi_1\varphi_{\mathrm{St}}$	-1
	real case			
$\chi_1 \neq \chi_2$	$L(\nu\,\mathrm{sgn}, \mathrm{sgn} \rtimes \nu^{-1/2})$	$(1,1)$, H	$\varphi_1 \oplus \mathrm{sgn}\,\varphi_1$	$+1$
	$\mathcal{D}^{\mathrm{hol}}(1,0)$	$(2,2)$, A	$\varphi_{\mathcal{D}(1)} \oplus \varphi_{\mathcal{D}(1)}$	-1
$\chi_1 = \chi_2$	$L(\nu, 1_{\mathbb{R}^\times} \rtimes \nu^{-1/2}\chi_1)$	$(0,0)$, D	$\chi_1\varphi_1 \oplus \chi_1\varphi_1$	$+1$
	$L(\nu^{1/2}\mathcal{D}(1) \rtimes \nu^{-1/2}\chi_1)$	$(1,-1)$, C	$\chi_1\varphi_1 \oplus \varphi_{\mathcal{D}(1)}$	-1

The real case

The calculation of the local packets for $F_v = \mathbb{R}$ is analogous to the non-archimedean case. The Weil representations $\tilde{\pi}^+_{\chi_v}$ have a vector of lowest (or highest) weight $1/2$ (or $-1/2$), and the $\tilde{\pi}^-_{\chi_v}$ have a vector of lowest (or highest) weight $3/2$ (or $-3/2$). Their theta liftings to the odd orthogonal tower can be determined using [26] and [12]. This leads to the packets summarized in Table 1. Alternatively, one can use Examples 1.4.2 and 1.4.3 of [1], in which the packets $\{\mathrm{H}, \mathrm{A}\}$ and $\{\mathrm{D}, \mathrm{C}\}$ have been determined.

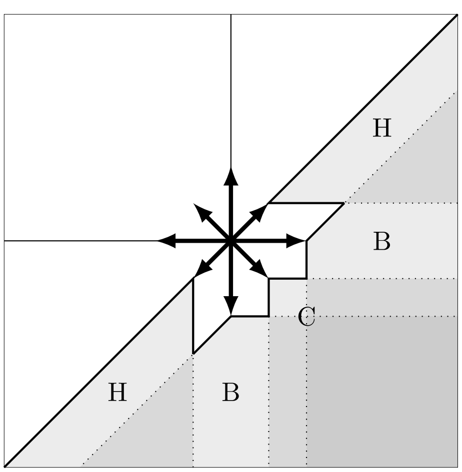
In the rest of this section we will explain the meaning of the "type" column in the archimedean case. We will use the conventions of [14] and [22] for K-types of representations of $\mathrm{GSp}(4,\mathbb{R})$. The symbol $\mathcal{D}(1)$ denotes the "lowest" discrete representation of $\mathrm{GL}(2,\mathbb{R})$, with minimal K-type 2 and trivial central character. Let σ be a quadratic character of $\mathbb{R}^\times$, which can only be the trivial character or the sign character sgn.

We will discuss two Borel induced representations of $\mathrm{PGSp}(4,\mathbb{R})$. By Theorem 11.2 of [14], the representation $\nu\,\mathrm{sgn} \times \mathrm{sgn} \rtimes \nu^{-1/2}\sigma$ has length 4. More precisely, its irreducible constituents are as follows:

i) $\mathcal{D}^{\mathrm{gen}}(1,0)$, the "large" (generic) limit of discrete series representation with minimal K-type $(2,0)$. Its K-types lie in the regions marked B in picture (20) below.

ii) $L(\nu^{1/2}\mathcal{D}(1) \rtimes \nu^{-1/2}\sigma)$, a non-tempered representation with minimal K-type $(1,-1)$. Its K-types can be determined from Lemma 6.1 and equation (10.25) of [14]; they lie in the "square" region C in picture (20). This representation appears with multiplicity 2 in $\nu\,\mathrm{sgn} \times \mathrm{sgn} \rtimes \nu^{-1/2}\sigma$.

iii) The Langlands quotient $L(\nu\,\mathrm{sgn} \times \mathrm{sgn} \rtimes \nu^{-1/2}\sigma)$, which has a minimal K-type at $(1,1)$. Its K-types can be determined from Lemma 6.1 of [14], subtracting the K-types of the other constituents from the K-types of the full induced representation; they lie in the disconnected "wedge" region H in picture (20). (This is the representation underlying holomorphic Siegel modular forms of weight 1. It is invariant under twisting by quadratic characters, so we may omit the σ.)



The representation $\nu\,\mathrm{sgn} \times \mathrm{sgn} \rtimes \nu^{-1/2}\sigma$ has four constituents: (20)

- $\mathcal{D}^{\mathrm{gen}}(1,0) = \mathrm{B}$.
- $L(\nu^{1/2}\mathcal{D}(1) \rtimes \nu^{-1/2}\sigma) = \mathrm{C}$ (appears with multiplicity 2).
- $L(\nu\,\mathrm{sgn} \times \mathrm{sgn} \rtimes \nu^{-1/2}) = \mathrm{H}$.

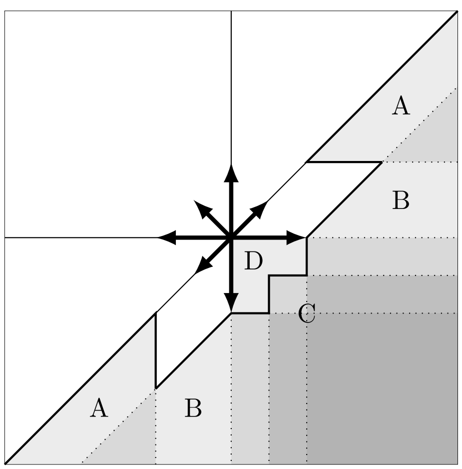
Next consider the representation $\nu \times 1_{\mathbb{R}^\times} \rtimes \nu^{-1/2}\sigma$ of $\mathrm{PGSp}(4,\mathbb{R})$. By Theorem 10.7 of [14], it has length 4. Its irreducible constituents are as follows:

i) $\mathcal{D}^{\mathrm{hol}}(1,0)$, the holomorphic limit of discrete series representation with minimal K-type $(2,2)$. (This is the representation underlying holomorphic Siegel modular forms of weight 2.) Its K-types lie in the disconnected "wedge" region A in picture (21) below.

ii) $\mathcal{D}^{\mathrm{gen}}(1,0)$, the same "large" limit of discrete series representation as above. Its K-types are contained in region B in picture (21).

iii) $L(\nu^{1/2}\mathcal{D}(1) \rtimes \nu^{-1/2}\sigma)$, the same non-tempered representation with minimal K-type $(1,-1)$ as above. Its K-types lie in the "square" region C in (21).

iv) The Langlands quotient $L(\nu \times 1_{\mathbb{R}^\times} \rtimes \nu^{-1/2}\sigma)$, which has a minimal K-type at $(0,0)$. Its K-types can be determined from Lemma 6.1 of [14], substracting the K-types of the other constituents from the K-types of the full induced representation; they lie in the fourth quadrant, indicated as region D in the picture below. More precisely, the multiplicity of the K-type (k_1,k_2) is 1 if k_1,k_2 are integers of the same parity with $k_1 \geq 0 \geq k_2$, and 0 otherwise.



The representation $\nu \times 1_{\mathbb{R}^\times} \rtimes \nu^{-1/2}\sigma$ has four constituents:

- $\mathcal{D}^{\mathrm{hol}}(1,0) = \mathrm{A}$.
- $\mathcal{D}^{\mathrm{gen}}(1,0) = \mathrm{B}$.
- $L(\nu^{1/2}\mathcal{D}(1) \rtimes \nu^{-1/2}\sigma) = \mathrm{C}$.
- $L(\nu, 1_{\mathbb{R}^\times} \rtimes \nu^{-1/2}\sigma) = \mathrm{D}$.

(21)

The type entry in Table 1 shows the minimal K-type of a representation and the region that contains all the K-types.

3 Local packets for type (P)

In this section we determine the local packets for Arthur parameters of Saito - Kurokawa type. As explained in [5] and [6], up to quadratic twists such Arthur packets are obtained as theta liftings of cusp forms on the metaplectic group $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ that are orthogonal to all global Weil representations $\tilde{\pi}_\chi^S$. By [26], such cusp forms on $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ are grouped into finite *Waldspurger packets* $\tilde{\Pi}_\mu$, which are their near-equivalence classes. They are parametrized by the unitary, cuspidal, automorphic representations $\mu \cong \otimes\mu_v$ of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character. The elements of $\tilde{\Pi}_\mu$ are tensor products $\tilde{\pi} \cong \otimes\tilde{\pi}_v$, where $\tilde{\pi}_v$ is taken from a local packet $\tilde{\Pi}_{\mu_v}$. The local packet has one element if μ_v is a principal series representation, and two elements if μ_v is square-integrable. Moreover, each local packet contains a base point assigned the sign $+1$, and the second representation in case of a two-element packet is assigned the sign -1. In order for $\otimes\tilde{\pi}_v$ to be an element of the global packet $\tilde{\Pi}_\mu$, the product of all local signs must equal $\varepsilon(1/2,\mu)$. In fact, if this condition is not satisfied, then $\otimes\tilde{\pi}_v$ does not define a representation of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$; see (1) on page 280 of [26].

3.1 Lemma. *Let $\mu = \otimes\mu_v$ be a unitary, cuspidal, automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character. Consider the Arthur parameter $\psi = (\mu \boxtimes \nu(1)) \boxplus (1 \boxtimes \nu(2))$ of Saito - Kurokawa type. As $\tilde{\pi}$ runs through the Waldspurger packet $\tilde{\Pi}_\mu$, their theta liftings to $\mathrm{SO}(5,\mathbb{A}) \cong \mathrm{PGSp}(4,\mathbb{A})$ run through the discrete automorphic elements of the Arthur packet Π_ψ.*

Proof. The proof is analogous to that of Lemma 2.1. By Lemme 49 of [26] or Lemma 7.2 of [15], the theta lifting of $\tilde{\pi} \in \tilde{\Pi}_\mu$ to $\mathrm{PGSp}(4,\mathbb{A})$ is near equivalent to any irreducible constituent of $|\cdot|^{1/2}\mu \rtimes |\cdot|^{-1/2}$. If $\tilde{\pi}$ is the global base point in the Waldspurger packet (assuming $\varepsilon(1/2,\mu) = 1$), then the lifting is isomorphic to the isobaric constituent of $|\cdot|^{1/2}\mu \rtimes |\cdot|^{-1/2}$. In this case, if $L(1/2,\mu) = 0$, the lifting is cuspidal by Proposition 24 of [26], and if $L(1/2,\mu) \neq 0$, it appears in the residual spectrum by Theorem 7.1 of [11]. If $\tilde{\pi}$ is not the base point, then it does not lift to $\mathrm{SO}(3,\mathbb{A}) \cong \mathrm{PGL}(2,\mathbb{A})$, and hence its lifting to $\mathrm{SO}(5,\mathbb{A}) \cong \mathrm{PGSp}(4,\mathbb{A})$ is cuspidal. We see that in all cases the lifting of $\tilde{\pi}$ to $\mathrm{PGSp}(4,\mathbb{A})$ is in the discrete spectrum. Hence the lifting must be contained in a packet Π_ψ. Looking at almost every place, we must have $\psi = (\mu\boxtimes\nu(1))\boxplus(1\boxtimes\nu(2))$ of type **(P)**.

Conversely, let π be a cuspidal element of Π_ψ. As we saw in Sect. 1, π is CAP to $|\cdot|^{1/2}\mu \rtimes |\cdot|^{-1/2}$. By Theorems 2.2 and 2.4 of [15], π is a theta lifting of a cusp form on $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ which is not a Weil representation. More precisely, by comparing local components almost everywhere, we see that π must be a lifting of an element of $\tilde{\Pi}_\mu$. ∎

It follows from Lemma 3.1 that the local Arthur packets Π_{ψ_v} for $\psi = (\chi \boxtimes \nu(2)) \boxplus (1 \boxtimes \nu(2))$ are the theta liftings of the local Waldspurger packets $\tilde{\Pi}_{\mu_v}$. These liftings have been calculated; see Table 2 of [20]. For arbitrary ψ of type **(P)**, the local packets are then obtained by twisting; see (13), (14), (15). We thus obtain the local packets summarized in Table 2. To determine the signs given in the last column of Table 2, we can argue as in the Borel case. Lemma 3.1, together with the structure of the Waldspurger packets on $\widetilde{\mathrm{SL}}(2,\mathbb{A})$, imply that the global Arthur packet Π_ψ is unstable. Hence, if a local packet has two elements, these two representations must be assigned different signs.

The real case

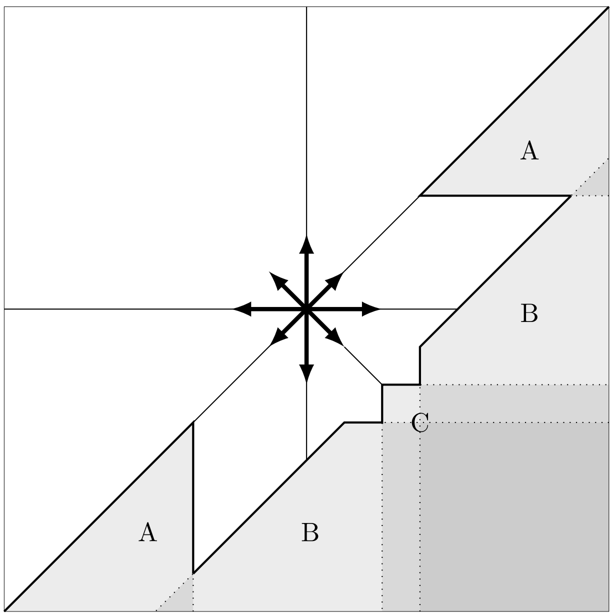
For an integer $\ell \geq 1$, let $\mathcal{D}(\ell)$ be the discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ with a lowest weight vector of weight $\ell + 1$ and central character $\mathrm{sgn}^{\ell+1}$. Let σ be a quadratic character of $\mathbb{R}^\times$. To determine the K-types of the representation $L(\nu^{1/2}\,\mathcal{D}(\ell), \nu^{-1/2}\sigma)$ appearing in Table 2, we consider $\nu^{1/2}\,\mathcal{D}(\ell) \rtimes \nu^{-1/2}\sigma$. We are only interested in odd ℓ, since only then does $\mathcal{D}(\ell)$ have trivial central character. We already determined the K-types of $L(\nu^{1/2}\mathcal{D}(1) \rtimes \nu^{-1/2}\sigma)$; see (20) and (21). Assuming $k \geq 3$ and setting $(p,t) = (k-1, k-2)$ in Theorem 10.1 of [14], we see that $\nu^{1/2}\,\mathcal{D}(2k-3) \rtimes \nu^{-1/2}\sigma$ has two irreducible constituents:

Table 2: Local Arthur packets Π_ψ of Saito - Kurokawa type (Siegel packets, type **(P)**). The local Arthur parameter ψ is determined by an irreducible, admissible, unitary representation μ of $\mathrm{PGL}(2,F)$, and a quadratic character σ of $F^\times$; see (3). The symbol ϕ stands for the L-parameter of μ. The parameters φ_1 and φ_{St} are defined in (18) and (19). For the K-types A and C, see (22).

μ	$\mathrm{GSp}(4,F)$	type	L-parameter	ϵ
		non-archimedean case		
$\chi \times \chi^{-1}$	$\chi\sigma 1_{\mathrm{GL}(2)} \rtimes \chi^{-1}$	IIb	$\chi \oplus \chi^{-1} \oplus \sigma\varphi_1$	$+1$
$\chi\mathrm{St}_{\mathrm{GL}(2)},\ \chi \neq \sigma$	$L(\nu^{1/2}\chi\sigma\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	Vb	$\chi\varphi_{\mathrm{St}} \oplus \sigma\varphi_1$	$+1$
	$\delta^*([\chi\sigma, \nu\chi\sigma], \nu^{-1/2}\sigma)$	Va*	$\chi\varphi_{\mathrm{St}} \oplus \sigma\varphi_{\mathrm{St}}$	-1
$\sigma\mathrm{St}_{\mathrm{GL}(2)}$	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	VIc	$\sigma\varphi_{\mathrm{St}} \oplus \sigma\varphi_1$	$+1$
	$\tau(T, \nu^{-1/2}\sigma)$	VIb	$\sigma\varphi_{\mathrm{St}} \oplus \sigma\varphi_{\mathrm{St}}$	-1
supercuspidal	$L(\nu^{1/2}\sigma\mu, \nu^{-1/2}\sigma)$	XIb	$\phi \oplus \sigma\varphi_1$	$+1$
	$\delta^*(\nu^{1/2}\sigma\mu, \nu^{-1/2}\sigma)$	XIa*	$\phi \oplus \sigma\varphi_{\mathrm{St}}$	-1
		real case		
$\chi \times \chi^{-1}$	$\chi\sigma 1_{\mathrm{GL}(2)} \rtimes \chi^{-1}$	IIb	$\chi \oplus \chi^{-1} \oplus \sigma\varphi_1$	$+1$
$\mathcal{D}(2k-3),\ k \geq 2$	$L(\nu^{1/2}\mathcal{D}(2k-3), \nu^{-1/2}\sigma)$	$(k-1, 1-k),\ \mathrm{C}$	$\varphi_{\mathcal{D}(2k-3)} \oplus \sigma\varphi_1$	$+1$
	$\mathcal{D}^{\mathrm{hol}}(k-1, k-2)$	$(k, k),\ \mathrm{A}$	$\varphi_{\mathcal{D}(2k-3)} \oplus \sigma\varphi_{\mathcal{D}(1)}$	-1

i) $\mathcal{D}^{\text{gen}}(k-1, 2-k)$, the "large" (generic) discrete series representation with minimal K-type $(k, 2-k)$. Its K-types lie in region B in picture (22) below.

ii) The Langlands quotient $L(\nu^{1/2}\mathcal{D}(2k-3) \rtimes \nu^{-1/2}\sigma)$, which has a minimal K-type at $(k-1, 1-k)$. Its K-types can be determined from Lemma 6.1 of [14], substracting the K-types of the other constituent from the K-types of the full induced representation; they are contained in region C in (22). This is the non-tempered cohomological representation mentioned in Proposition 7.7 of [13].



The representation $\nu^{1/2}\mathcal{D}(2k-3) \rtimes \nu^{-1/2}\sigma$ has two constituents:

- $\mathcal{D}^{\text{gen}}(k-1, -k+2) = \text{B}$.
- $L(\nu^{1/2}\mathcal{D}(2k-3), \nu^{-1/2}\sigma) = \text{C}$.

The diagram shows the case $k = 3$. Shown is also the representation (22)

- $\mathcal{D}^{\text{hol}}(k-1, k-2) = \text{A}$.

with minimal K-type at (k, k).

The representation $\mathcal{D}^{\text{hol}}(k-1, k-2)$ appearing in Table 2 is the holomorphic discrete series representation of $\text{PGSp}(4, \mathbb{R})$ with scalar minimal K-type (k, k). It is the representation underlying Siegel modular forms of weight k. Its K-types are well known; see Sect. 2.2 of [22]. In (22), they lie in the disconnected "wedge" region A.

4 Local packets for type (Q)

Let $\psi = \mu \boxtimes \nu(2)$ be an Arthur parameter of Soudry type. Recall that $\mu = \otimes\mu_v$ is a self-dual, unitary, cuspidal automorphic representation of $\text{GL}(2, \mathbb{A})$ with non-trivial central character $\xi = \otimes\xi_v$. The central character determines a quadratic extension E of F. The representation μ is obtained by automorphic induction from a non-Galois-invariant character $\theta = \otimes\theta_w$ of $\mathbb{A}_E^\times$.

By [25], [8] or [16], the elements of the packet Π_ψ can also be obtained as theta liftings. Since we will not need the details of this construction, we only explain it briefly. The theta correspondence is one with similitudes, between $\text{GSp}(4, \mathbb{A})$ and $\text{GO}(2, \mathbb{A}_E)$. Here, E is viewed as a 2-dimensional quadratic space over F, endowed with the norm form. Locally, $\text{GSO}(2, E_w) = E_w^\times$, and $\text{GO}(2, E_w) = \langle\tau_w\rangle \ltimes E_w^\times$, where τ_w is the non-trivial Galois element of E_w/F_v. If the character θ_w is not Galois-invariant, $\theta_w^+ := \text{ind}_{\text{GSO}(2,E_w)}^{\text{GO}(2,E_w)}(\theta_w)$ is an irreducible 2-dimensional representation of $\text{GO}(2, E_w)$. Otherwise, θ_w admits two extensions $\theta_w^\pm$ to $\text{GO}(2, E_w)$. In the non-Galois-invariant case, the local packet Π_{ψ_v} has one element, namely the theta lift of θ_w^+, and in the Galois-invariant case, Π_{ψ_v} has two elements, namely the lifts of $\theta_w^{\pm}$.

Instead of calculating these theta lifts, which is the approach taken in [8], we observe the following. If the packet has only one element, it must be the base point already determined in Sect. 1; see (11). Otherwise, i.e. in the Galois-invariant cases, the parameters (6) are actually of Howe - Piatetski-Shapiro type. More precisely, for $\xi_v \neq 1$ (i.e., E_w is a field) and $\theta_w = \sigma_v \circ N_{E_w/F_v}$, conjugation by the matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & \\ -i & i & & \\ & & -i & i \\ & & 1 & 1 \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{C}) \tag{23}$$

transforms the parameter (6) into the parameter (2) with $\chi_{1,v} = \sigma_v$ and $\chi_{2,v} = \sigma_v \xi_v$. Similarly, for $\xi_v = 1$ (i.e., $E_w = F_v \times F_v$) and $\theta_w = (\sigma_v, \sigma_v)$, conjugation by A transforms the parameter (6) into the parameter (2) with $\chi_{1,v} = \chi_{2,v} = \sigma_v$. Hence the Klingen packets in these cases are the same as Borel packets, which we already determined in Table 1. More precisely, for $\xi_v \neq 1$ and $\theta_w = \sigma_v \circ N_{E_w/F_v}$, the packet is of type $\{\mathrm{Vd}, \mathrm{Va}^*\}$, and for $\xi_v = 1$ and $\theta_w = (\sigma_v, \sigma_v)$, the packet is of type $\{\mathrm{VId}, \mathrm{VIc}\}$.

Table 3 summarizes the local Klingen packets.

Local archimedean packets

The determination of the local Arthur packets for $F = \mathbb{R}$ is analogous to the non-archimedean case. The base points have L-parameter (6), and the two-element packets coincide with the archimedean Borel packets in Table 1. Our only goal in this subsection is to understand the K-types of the representation $L(\nu\,\mathrm{sgn}, \nu^{-1/2}\mathcal{D}(\ell))$, which is the single element in the Klingen packet for $\xi \neq 1$ and θ not Galois-invariant. Here, for an integer $\ell \geq 1$, the symbol $\mathcal{D}(\ell)$ denotes the discrete series representation of $\mathrm{GL}(2, \mathbb{R})$ with a lowest weight vector of weight $\ell + 1$ and central character $\mathrm{sgn}^{\ell+1}$.

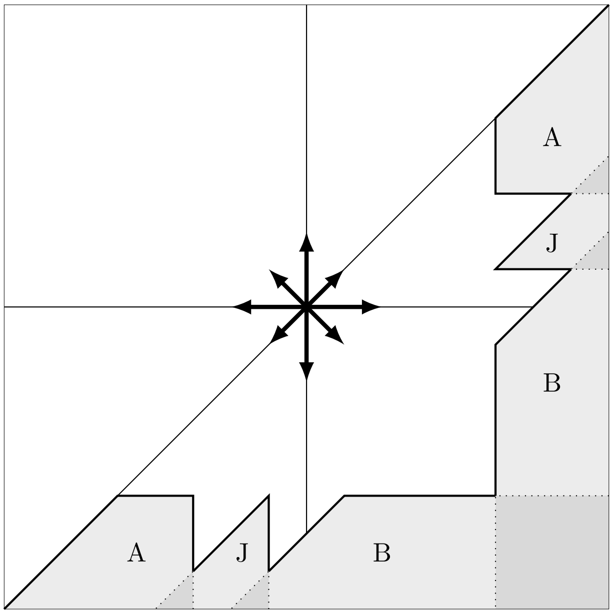
Assume that $\ell \geq 2$. By Theorem 10.1 of [14], the representation $\nu\,\mathrm{sgn} \rtimes \nu^{-1/2}\mathcal{D}(\ell)$ has three irreducible subquotients:

i) $\mathcal{D}^{\mathrm{hol}}(\ell, 1)$, the holomorphic discrete series representation with minimal K-type $(\ell+1, 3)$. (This is the representation underlying vector-valued holomorphic Siegel modular forms of weight $\det^3 \mathrm{sym}^{\ell-2}$.) Its K-types lie in region A in the picture (24).

ii) $\mathcal{D}^{\mathrm{gen}}(\ell, -1)$, the "large" (generic) discrete series representation with minimal K-type $(\ell+1, -1)$. Its K-types lie in region B in the picture (24).

iii) The Langlands quotient $L(\nu\,\mathrm{sgn}, \nu^{-1/2}\mathcal{D}(\ell))$, which has a minimal K-type at $(\ell+1, 1)$. (This is the representation underlying vector-valued holomorphic Siegel modular forms of weight $\det^1 \mathrm{sym}^{\ell}$.) Its K-types can be determined from Lemma 6.1 of [14], substracting the K-types of the other constituents from the K-types of the full induced representation; they lie in region J in the picture (24).

Note that $\nu\,\mathrm{sgn} \rtimes \nu^{-1/2}\mathcal{D}(\ell)$ has central character sgn^{ℓ}, so only the case of even ℓ will be relevant for us.



The representation $\nu\,\mathrm{sgn} \rtimes \nu^{-1/2}\mathcal{D}(\ell)$ has three constituents:

- $\mathcal{D}^{\mathrm{hol}}(\ell, 1) = \mathrm{A}$.
- $\mathcal{D}^{\mathrm{gen}}(\ell, -1) = \mathrm{B}$. (24)
- $L(\nu\,\mathrm{sgn}, \nu^{-1/2}\mathcal{D}(\ell)) = \mathrm{J}$.

The diagram shows the case $\ell = 4$.

The limit case $\ell = 1$ is handled by Theorem 10.4 ii) of [14]. In this case $\nu\,\mathrm{sgn} \rtimes \nu^{-1/2}\mathcal{D}(1)$ has only two irreducible constituents. This representation is not actually relevant for us since it does not have trivial central character.

5 Paramodular forms

In this section we assume that the ground field is $\mathbb{Q}$. We switch to the "classical" version of GSp(4), defined with the symplectic form $\left[\begin{smallmatrix} & 1_2 \\ -1_2 & \end{smallmatrix}\right]$. For a congruence subgroup Γ of $\mathrm{Sp}(4, \mathbb{Q})$ and non-negative integers k and j, let $S_{k,j}(\Gamma)$ be the space of Siegel modular cusp forms of weight $\det^k \mathrm{sym}^j$ with respect to Γ; see Sect. 2.1 of [23] for the precise definition. For $j = 0$ we write $S_k(\Gamma)$; this is the usual space of scalar-valued cusp forms of weight k.

For a positive integer N, the paramodular group of level N is defined as

$$K(N) = \mathrm{Sp}(4, \mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}. \qquad (25)$$

We are primarily interested in the spaces $S_{k,j}(K(N))$. As explained in [17], these spaces admit a theory of new- and oldforms.

We see from the archimedean part of Table 1 that it is possible to construct holomorphic Siegel modular forms of weight 1 and 2 from Borel CAP representations. Similarly, it follows from the archimedean part of Table 3 that it is possible to construct holomorphic Siegel modular forms of weight 1 and 2, and also certain vector-valued holomorphic Siegel modular forms of weight $\det^1 \mathrm{sym}^\ell$, from Klingen CAP representations. However, as the following argument shows, none of these can be paramodular.

5.1 Proposition. *No representation in an Arthur packet of type* **(B)** *(Howe - Piatetski-Shapiro type) or* **(Q)** *(Soudry type) is paramodular at every finite place.*

Table 3: Local Arthur packets Π_ψ of Soudry type (Klingen packets, type **(Q)**). The local Arthur parameter ψ is determined by a quadratic character ξ of $F^\times$ and a character θ of $E^\times$, where E is the quadratic extension determined by ξ; see (6). In the row for IXb, ϕ is the L-parameter of $\mu = \mathcal{AI}_{E,\theta}$, a supercuspidal representation of $\mathrm{GL}(2,F)$. In the row for J, ϕ is the L-parameter of $\mathcal{D}(\ell)$, a discrete series representation of $\mathrm{GL}(2,\mathbb{R})$. The positive integer ℓ needs to be even in order for the $\mathrm{GSp}(4,\mathbb{R})$ representation to have trivial central character. For the K-type H, see (20); for A, C and D, see (21).

$E \leftrightarrow \xi$	θ	$\mathrm{GSp}(4,F)$	type	L-parameter
		non-archimedean case		
$\xi \neq 1$	not Galois-invariant	$L(\nu\xi, \nu^{-1/2}\mu)$	IXb	$\phi \otimes \varphi_1$
	$\sigma \circ N_{E/F}$	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	Vd	$(\sigma \oplus \sigma\xi) \otimes \varphi_1$
		$\delta^*([\xi, \nu\xi], \nu^{-1/2}\sigma)$	Va*	$(\sigma \oplus \sigma\xi) \otimes \varphi_{\mathrm{St}}$
$\xi = 1$	$(\theta_1, \theta_2),\ \theta_1 \neq \theta_2$	$\theta_1\theta_2^{-1} \rtimes \theta_2 1_{\mathrm{GSp}(2)}$	IIIb	$(\theta_1 \oplus \theta_2) \otimes \varphi_1$
	(σ, σ)	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	VId	$(\sigma \oplus \sigma) \otimes \varphi_1$
		$L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma)$	VIc	$\sigma\varphi_1 \oplus \sigma\varphi_{\mathrm{St}}$
		real case		
$\xi \neq 1$	not Galois-invariant	$L(\nu\xi, \nu^{-1/2}\mathcal{D}(\ell))$	$(\ell+1, 1)$, J	$\phi \otimes \varphi_1$
	$\sigma \circ N_{E/F}$	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$(1, 1)$, H	$(\sigma \oplus \sigma\xi) \otimes \varphi_1$
		$\mathcal{D}^{\mathrm{hol}}(1, 0)$	$(2, 2)$, A	$(\sigma \oplus \sigma\xi) \otimes \varphi_{\mathcal{D}(1)}$
$\xi = 1$	$(\theta_1, \theta_2),\ \theta_1 \neq \theta_2$	$\theta_1\theta_2^{-1} \rtimes \theta_2 1_{\mathrm{GSp}(2)}$	IIIb	$(\theta_1 \oplus \theta_2) \otimes \varphi_1$
	(σ, σ)	$L(\nu, 1_{\mathbb{R}^\times} \rtimes \nu^{-1/2}\sigma)$	$(0, 0)$, D	$(\sigma \oplus \sigma) \otimes \varphi_1$
		$L(\nu^{1/2}\,\mathcal{D}(1) \rtimes \nu^{-1/2}\sigma)$	$(1, -1)$, C	$\sigma\varphi_1 \oplus \sigma\varphi_{\mathcal{D}(1)}$

Proof. Let χ_1, χ_2 be distinct quadratic Hecke characters, and consider the corresponding Arthur packet of Howe - Piatetski-Shapiro type. We factor $\chi_i = \otimes \chi_{i,v}$. Since $\chi_1\chi_2$ is a non-trivial character, there exists a finite place v such that $\chi_{1,v}\chi_{2,v}$ is ramified. By Table 1, the local Arthur packet at v is of type $\{\mathrm{Vd}, \mathrm{Va}^*\}$. Neither of the two representations is paramodular: Vd is not, because $\chi_1\chi_2$ is ramified, and Va^* is not because it is a non-generic supercuspidal; see Theorem 3.4.3 of [18].

Let μ be a self-dual, unitary, cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with non-trivial central character, determining an Arthur packet of Soudry type. Let $E/\mathbb{Q}$ be the quadratic extension corresponding to the central character ξ of μ, and let θ be a character of $\mathbb{A}_E^\times$ such that $\mu = \mathcal{AI}_{E/\mathbb{Q}}(\theta)$. We factor $\xi = \otimes \xi_v$. Since ξ is non-trivial, there exists a finite place v of E for which ξ_v is ramified. By Table 3, the local Arthur packet is either of type IXb, or of type $\{\mathrm{Vd}, \mathrm{Va}^*\}$. Again by Theorem 3.4.3 of [18], none of these representations contains paramodular vectors. ■

Note that the non-existence for type **(B)** in this proposition is not based on the instability of the global Arthur packets.

Now consider a global Arthur parameter ψ of Saito - Kurokawa type, given by a pair (μ, σ), where μ is a unitary, cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character, and σ is a quadratic Hecke character. The representations $\pi = \otimes \pi_v$ in the global packet Π_ψ are obtained by choosing one element π_v from each local packet Π_{ψ_v}, with π_v being the base point almost everywhere. By Arthur's multiplicity formula, π will appear in the discrete automorphic spectrum if and only if

$$\epsilon(\pi) := \prod_v \epsilon(\pi_v) = \varepsilon(1/2, \sigma \otimes \mu). \tag{26}$$

If π appears, then it does so with multiplicity one.

5.2 Proposition. *Consider a global Arthur packet Π_ψ of Saito - Kurokawa type, parametrized by a pair (μ, σ), where $\mu = \otimes \mu_v$ is a unitary, cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ with trivial central character, and σ is a quadratic Hecke character.*

- *i) If σ is non-trivial, then no representation in the packet Π_ψ is paramodular at every finite place.*
- *ii) If σ is trivial and μ_∞ is a discrete series representation, then there exists a unique representation $\pi = \otimes \pi_v$ in the packet Π_ψ that is paramodular at every finite place and appears in the discrete automorphic spectrum. For each finite place, π_v is the base point in the packet Π_{ψ_v}.*

Proof. Assume that $\sigma = \otimes \sigma_v$ is non-trivial. Then there exists a finite place v for which σ_v is ramified. A look at Table A.12 of [18] shows that none of the non-archimedean representations listed in Table 2 is paramodular (for ramified quadratic character). Hence none of the representations in the global Arthur packet is paramodular everywhere.

Assume that σ is trivial. Then Table A.12 of [18] shows that precisely the base point in each local packet is paramodular. Let π_v be this base point, for each finite v. Assume in addition that μ_∞ is a discrete series representation, so that the archimedean local packet has two elements.

Choose π_∞ from this packet in such a way that (26) is satisfied. Then $\pi = \otimes\pi_v$ is the unique element in the global Arthur packet which appears in the discrete spectrum and is paramodular at every finite place. ∎

We can now deduce the following result on paramodular Saito-Kurokawa liftings. A version for square-free levels was proven in [21]. The existence of the lifting can also be proven by combining the isomorphism from [24] with the Gritsenko lifting from [7]. We mention that in [9], starting from Jacobi forms, Saito-Kurokawa liftings for arbitrary levels are constructed, not with respect to the paramodular group, but with respect to the Siegel congruence subgroup $\Gamma_0^{(2)}(N)$.

5.3 Theorem. *Let $N \geq 1$ and $k \geq 2$ be integers. Let $f \in S_{2k-2}(\Gamma_0(N))$ be an eigenform and a newform. Assume that the sign in the functional equation of $L(s,f)$ is -1. Then there exists a paramodular form $F \in S_k(K(N))$, unique up to multiples, with the following properties.*

i) F is an eigenform for all good Hecke operators. The complete spin L-function of F is given by

$$L(s,F) = \frac{1}{4\pi}(s-1/2)L(s,f)Z(s-1/2)Z(s+1/2). \tag{27}$$

Here, $Z(s)$ is the completed Riemann zeta function.

ii) F is a newform in the sense of [17].

iii) For each prime p, the Atkin-Lehner eigenvalue of F at p coincides with the Atkin-Lehner eigenvalue of f at p.

iv) Suppose that $G \in S_\ell(K(M))$ is an eigenform and newform which has the same Hecke eigenvalues as F almost everywhere. Then $\ell = k$ and $M = N$, and G is a multiple of F.

v) The adelization of F generates an irreducible, cuspidal, automorphic representation $\pi \cong \otimes\pi_v$ of $\mathrm{GSp}(4,\mathbb{A})$. If μ is the cuspidal, automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ generated by f, then π lies in the Arthur packet Π_ψ, where $\psi = (\mu \boxtimes 1) \boxplus (1 \boxtimes \nu(2))$ is of Saito-Kurokawa type.

Proof. Let μ be the cuspidal, automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ generated by f. Then $\psi = (\mu \boxtimes 1) \boxplus (1 \boxtimes \nu(2))$ is an Arthur parameter of Saito-Kurokawa type. By Proposition 5.2, there exists a unique representation $\pi = \otimes\pi_v$ in Π_ψ which appears in the discrete spectrum and is paramodular at every finite place. Our hypothesis on the sign in the functional equation means that $\varepsilon(1/2,\mu) = -1$. For each finite place, π_v is the base point in the packet Π_{ψ_v}, which has the sign $\epsilon(\pi_v) = 1$ attached to it. By (26), we must have $\epsilon(\pi_\infty) = -1$. Hence $\pi_\infty = \mathcal{D}^{\mathrm{hol}}(k-1,k-2)$ by Table 2. Lemma 1.2 implies that π is cuspidal. It follows that we can extract from π a holomorphic cuspidal paramodular newform F of weight k, as explained (in greater generality) in Sect. 4.2 of [22]. Since F originates from the irreducible automorphic representation π, its adelization (as defined in (4) of [17]) generates π, proving the first statement of v).

By properties of the local paramodular theory, the level of F will be $\prod p^{a(\pi_p)}$, where $a(\pi_p)$ is the conductor of the L-parameter of π_p; see Theorem 7.5.9 of [18]. A look at the L-parameter column of Table 2 shows that $a(\pi_p) = a(\mu_p)$. It follows that $F \in S_k(K(N))$.

By Theorem 7.5.9 of [18], the Atkin-Lehner eigenvalue of F at p coincides with $\varepsilon(1/2,\pi_p)$. A look at the L-parameter column of Table 2 shows that $\varepsilon(1/2,\pi_p)=\varepsilon(1/2,\mu_p)$. This proves iii), since $\varepsilon(1/2,\mu_p)$ coincides with the Atkin-Lehner eigenvalue of f at p.

Since $L(s,\varphi_1)=((1-p^{-s-1/2})(1-p^{-s+1/2}))^{-1}$, the L-parameter column of Table 2 shows that $L(s,F)=L(s,f)Z(s-1/2)Z(s+1/2)$ for the incomplete L-function that incorporates only the finite places. At the archimedean place, the L-factor of $\mathcal{D}^{\rm hol}(k-1,k-2)$ is

$$\Gamma_{\mathbb{C}}(s+k-3/2)\Gamma_{\mathbb{C}}(s+1/2),$$

where $\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)$; see Table 5 of [22]. The L-factor of $\mathcal{D}(2k-3)$ is $\Gamma_{\mathbb{C}}(s+k-3/2)$. The archimedean Euler factor for $Z(s)$ is $\pi^{-s/2}\Gamma(s/2)$. Elementary properties of the Γ-function then explain the factor $\frac{1}{4\pi}(s-1/2)$ in (27).

We now proved i), ii), iii) and v). Let G be as in iv). Let π' be the cuspidal representation generated by the adelization of G. It decomposes as a finite direct sum $\pi_1\oplus\ldots\oplus\pi_n$ of irreducible, cuspidal, automorphic representations of ${\rm GSp}(4,\mathbb{A})$ with trivial central character. By our hypothesis on the Hecke eigenvalues, each π_i is near equivalent to the representation π generated by F, and hence lies in the packet Π_ψ. Moreover, since G is a paramodular form, each π_i is paramodular at every finite place. Using Proposition 5.2, it follows that $n=1$ and $\pi'=\pi$. Since F and G are both global newforms, their adelizations are pure tensors of local newforms at every finite place. Hence F and G are multiples of each other and $M=N$. Looking at the archimedean place, we see that $\ell=k$. ■

In Sect. 2.1 of [23] the *type* of a Siegel eigenform F was defined to be the type of the Arthur packet containing (the adelization of) F. The subspace of $S_{k,j}(\Gamma)$ spanned by all eigenforms of type **(G)** is denoted by $S_{k,j}(\Gamma)_{\mathbf{(G)}}$, and similarly for the other types.

5.4 Corollary. *Let k and N be positive integers, and j a non-negative integer. Then*

$$S_{k,j}(K(N))=S_{k,j}(K(N))_{\mathbf{(G)}}\oplus S_{k,j}(K(N))_{\mathbf{(P)}}. \tag{28}$$

If $j>0$, then $S_{k,j}(K(N))_{\mathbf{(P)}}=0$.

Proof. By Lemma 2.5 of [23], the space $S_{k,j}(K(N))_{\mathbf{(Y)}}$ is zero. By Proposition 5.1, the spaces $S_{k,j}(K(N))_{\mathbf{(B)}}$ and $S_{k,j}(K(N))_{\mathbf{(Q)}}$ are also zero. This proves (28). Table 2 shows that if a packet of type **(P)** contains a lowest weight representation at the archimedean place, then this representation must have scalar minimal K-type. Hence one cannot construct vector-valued holomorphic cusp forms from packets of type **(P)**. ■

The multiplicity one theorem for paramodular forms, Theorem 2.6 of [23], had a hypothesis that forms be of type **(G)**. This assumption can now be removed:

5.5 Theorem. *Let N,N_1,N_2 and k,k_1,k_2 be positive integers, and j,j_1,j_2 be non-negative integers.*

i) Assume that $F\in S_{k,j}(K(N))$ is an eigenform for the unramified local Hecke algebra $\mathcal{H}_p$ for almost all p not dividing N. Then F is an eigenform for $\mathcal{H}_p$ for <u>all</u> $p\nmid N$. The cuspidal, automorphic representation π of $G(\mathbb{A})$ generated by the adelization of F is irreducible. The conductor of π divides N, with equality if and only if F is a newform.

ii) Let $F_i \in S^{\text{new}}_{k_i,j_i}(K(N_i))$, $i = 1, 2$, be two eigenforms. Assume that for almost all primes p the Hecke eigenvalues of F_1 and F_2 coincide. Then $(k_1, j_1) = (k_2, j_2)$, $N_1 = N_2$, and F_1 is a multiple of F_2.

Proof. Making use of Corollary 5.4 and Proposition 5.2 ii), the proof is similar to that of Theorem 2.6 of [23]. ■

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