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On deformations of crossed products

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Abstract

Let $A * \Gamma$ be a crossed product algebra, where A is semisimple, finitely generated over its center and Γ is a finite group. We give a necessary and sufficient condition in terms of the outer action of Γ on A for the existence of a multi-parametric semisimple deformation of the form $A((t_1, \ldots, t_n)) * \Gamma$ (with the induced outer action). The main tool in the proof is the solution of the so-called twisting problem. We also give an example which shows that the condition is not sufficient if one drops the condition on the finite generation of A over its center.

MSC: 16K20

1. Introduction

Let $R = A * \Gamma$ be a crossed product algebra, where A is an artinian ring, finitely generated over its center, and Γ a finite group. The purpose of this article is to provide a necessary and sufficient condition for the existence of a semisimple homogeneous deformation of R. By this we mean a deformation of R which also has a crossed product structure compatible with the structure of R (see Definition 1.5 below). It is shown that such a deformation must essentially be given by purely inseparable extensions of the centers of the simple components of A. We show that the necessary and sufficient condition mentioned above is not sufficient if one drops the requirement on A of being finitely generated over its center.

Since our task is to construct semisimple crossed products, it is natural to ask in general when a (given) crossed product $A * \Gamma$ (Γ finite) is semisimple artinian. Of course, one may answer this question in more than one way. A satisfactory answer is given in [2], which is based on results from [4,10,12]. Let us sketch briefly the answer given there.

To begin with, since $A * \Gamma$ is free over A, $A * \Gamma$ cannot be semisimple artinian unless A itself is semisimple artinian. Therefore, we assume for the rest of the paper that the base ring A is semisimple artinian.

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Recall that a crossed product $A * \Gamma$ induces an action of Γ on the set of simple components $\Lambda = \{A_i\}_{i \in I}$ of A. Let $\{A_{i_j}\}_{j \in J}$ be a set of representatives for the orbits in Λ and let Γ_{i_j} be the stabilizer of A_{i_j} in Γ. Then $A * \Gamma \cong \prod_{j \in J} M_{n_j}(A_{i_j} * \Gamma_{i_j})$ with $n_j = [\Gamma : \Gamma_{i_j}]$. The first step is

Proposition 1.1 (*[10], Theorem 7.5*). *The crossed product $A * \Gamma$ is semisimple if and only if the induced crossed products $A_{i_j} * \Gamma_{i_j}$ are semisimple for all j's.*

Next, consider a crossed product $A * \Gamma$, where A is simple artinian. Let $\alpha : \Gamma \to \mathrm{Out}(A)$ be the induced homomorphism (see (2.1) below) and denote its kernel by H. One shows that after a diagonal change of base elements, the sub-crossed product $A * H$ is isomorphic to $A \otimes_{Z(A)} Z(A)^f H$, where $Z(A)^f H$ is a twisted group algebra over the center $Z(A)$, and f is a 2-cocycle of H with values in $Z(A)^*$ (see [2], Section 3). The second step is

Proposition 1.2 (*[2], Theorem 3.3*). *The following are equivalent:*

(1) *$A * \Gamma$ is semisimple*
(2) *$A * H$ is semisimple*
(3) *$Z(A)^f H$ is semisimple.*

The last part in the analysis is to determine when a twisted group algebra $K^f H$ is semisimple, where K is a field. In the non-modular case, namely if $\mathrm{ord}(H) \in K^*$, the twisted group algebra is always semisimple. In the modular case the answer is

Proposition 1.3 (*[3], Theorem 2*). *Assume* $\mathrm{char}(K) = p$ *and let P be a p-Sylow subgroup of H. Then $K^f H$ is semisimple if and only if*

(1) *$P = Z_{p^{\beta_1}} \times \cdots \times Z_{p^{\beta_r}} = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_r \rangle$ is abelian and has a normal complement in H.*
(2) *Let $(u_{\sigma_1}, \ldots, u_{\sigma_r})$ be elements in $K^f H$ representing $(\sigma_1, \ldots, \sigma_r)$ and let $u_{\sigma_i}^{p^{\beta_i}} = \alpha_i \in K^*$. Then the elements $\alpha_1, \ldots, \alpha_r$ are p-independent over the subfield K^p (see Definition 2.3 below). Equivalently, the twisted group algebra $K^f P$ is a purely inseparable field extension of K.*

From Propositions 1.1–1.3 we easily obtain a necessary condition (which depends only on the outer action $\alpha : \Gamma \to \mathrm{Out}(A)$) for the semisimplicity of a crossed product $R = A * \Gamma$:

Corollary 1.4. *Let $\{A_{i_j}\}_j$ be a set of simple representatives of the orbits and Γ_{i_j} be their stabilizers as in Proposition 1.1. Let $H_{i_j} = \ker(\alpha_{i_j} : \Gamma_{i_j} \to \mathrm{Out}(A_{i_j}))$ and let P_{i_j} be p_i-Sylow subgroups of H_{i_j} (in case* $\mathrm{char}(Z(A_{i_j})) = p_i > 0$*). Then $R = A * \Gamma$ is semisimple only if for every i_j such that $p_i > 0$*

(A) *P_{i_j} is abelian with normal complement in H_{i_j}*
(B) $\mathrm{rank}(P_{i_j}) \leq$ *p_i-degree of $Z(A_{i_j})$ (see Definition 2.3 below).*

Let R be a finite dimensional K-algebra. Let $K((t_1, \ldots, t_n))$ be the field of power series on n variables. Then an n-parameter deformation of R is an associative $K((t_1, \ldots, t_n))$-algebra R', whose structure as a $K((t_1, \ldots, t_n))$-vector space is the same as $R \otimes_K K((t_1, \ldots, t_n))$, such that the multiplication in R' deforms the multiplication in the algebra $R((t_1, \ldots, t_n)) = R \otimes_K K((t_1, \ldots, t_n))$:

$$x_1 * x_2 = x_1 \cdot x_2 + \sum \Psi_{i_1,\ldots,i_n}(x_1, x_2) t_1^{i_1} \ldots t_n^{i_n}, \quad x_1, x_2 \in R'. \tag{1.1}$$

Here $x_1 \cdot x_2$ is the original multiplication, and the sum runs over all $i_1, \ldots, i_n \geq 0$ which are not all zeros. The functions $\Psi_{i_1,\ldots,i_n}$ are bilinear and satisfy associativity conditions; see [9].

We shall be interested in deformations of crossed products which preserve the original group graded structure:

Definition 1.5. Let $R = A * \Gamma$ be a crossed product. We say that a crossed product $R' = A((t_1, \ldots, t_n)) * \Gamma$ is an n-parameter homogeneous deformation of R if the outer action α' of Γ on $A((t_1, \ldots, t_n))$ (induced from the crossed product structure on R') is the "same" as the outer action α on A induced from $R = A * \Gamma$, that is, $t_1, \ldots, t_n$ are central in R' and $i \circ \alpha = \alpha'$, where $i : \mathrm{Out}(A) \to \mathrm{Out}(A((t_1, \ldots, t_n)))$ is the natural embedding.

Clearly, by Corollary 1.4, $R = A * \Gamma$ admits a semisimple homogeneous deformation only if for all $i \in I$, P_{i_j} is abelian and has a normal complement in H_{i_j}.

Our main result is to show that the condition above is sufficient provided that A is finitely generated over its center.

Theorem 1.6. *Let $R = A * \Gamma$ be a crossed product, where Γ is finite and A is semisimple finitely generated over its center. Assume (with the above notation) that each P_{i_j} is abelian and has a normal complement in H_{i_j}. Then there is a semisimple n-parameter homogeneous deformation $R' = A((t_1, \ldots, t_n)) * \Gamma$ for some integer $n \geq 0$.*

It is natural to ask how many parameters are needed for such deformation of $A * \Gamma$. To answer this question we let A_{i_j}, Γ_{i_j}, H_{i_j} and P_{i_j} be as above. Assume P_{i_j} is abelian with normal complement in H_{i_j}. By Theorem 1.6 there is an n-parameter homogeneous semisimple deformation for some $n \geq 0$. Let $n_0 = n(A * \Gamma)$ be the (minimal) number of parameters needed to produce such a deformation.

Theorem 1.7. *Let $p(i)$ be the p_i-degree of $Z(A_{i_j})$, then the minimal number of parameters is given by $n_0 = \max_{i \in I}\{1, \operatorname{rank}(P_{i_j}) - p(i)\}$.*

Theorems 1.6 and 1.7 are derived from a positive answer to the *Twisting Problem* given in Section 3. The discussion in this section is based on the solution of the Twisting Problem over fields in [3,4].

The question of under which conditions a given crossed product $A * \Gamma$ (Γ finite) is semisimple may be viewed as a special case of the following problem: find necessary and sufficient conditions which imply that a given crossed product $A * \Gamma$ is *relative semisimple*, i.e. any $A * \Gamma$-module M which is projective over A is projective also over $A * \Gamma$ (see, e.g., [1,6,8]). We also refer the reader to [14] for a thorough treatment of the question of when a given crossed product $A * \Gamma$ (A semiprime, Γ finite) is semiprime.

2. Preliminaries and a main lemma

The proof of Theorem 1.6 is based on a detailed analysis of the purely inseparable extensions contained in $A * \Gamma$. Derivation maps play an important role in this analysis, and in particular, the Jacobian map. In this section we recall some preliminaries and prove a general Lemma 2.7 which will be essential in the proof of Theorem 1.6.

2.1

Recall that a Γ-graded algebra $R_{(\Gamma)} = \oplus_{g \in \Gamma} R_g$ with base ring $A = R_e$ is a crossed product if the component R_g admits a unit u_g for every $g \in \Gamma$. Then the set $\{u_g : g \in \Gamma\}$ is a basis of $R_{(\Gamma)}$ as a right A-module and there are maps

$$\beta : \Gamma \to \operatorname{Aut}(A), \qquad f : \Gamma \times \Gamma \to A^*$$

called the action and twisting respectively (A^* denotes the group of units of A). They are defined by

$$a u_g = u_g a^{\beta(g)}, \qquad u_g u_h = u_{gh} f(g, h)$$

for every $g, h \in \Gamma$ and $a \in A$. We usually simplify the notation and write a^g for $a^{\beta(g)}$. The action and twisting satisfy the following conditions:

$$f(g_1 g_2, g_3) f(g_1, g_2)^{\beta(g_3)} = f(g_1, g_2 g_3) f(g_2, g_3), \qquad \beta(g_1 g_2)\iota_{f(g_1, g_2)} = \beta(g_1)\beta(g_2) \tag{2.1}$$

for every $g_1, g_2, g_3 \in \Gamma$ (where ι_a means conjugation by $a \in A^*$). By (2.1), the map β induces a homomorphism $\alpha : \Gamma \to \operatorname{Out}(A)$ which restricts to an action of Γ on $Z(A)$. The action and twisting above depend on the choice of the set $\{u_g : g \in \Gamma\}$. Another choice of basis $\{v_g : g \in \Gamma\}$ yields an action β' and a twisting f'. Since the bases satisfy $v_g = \lambda_g u_g$, $\lambda_g \in A^*$, $g \in \Gamma$, it follows that $\beta'(g)$ differs from $\beta(g)$ by inner automorphisms for every $g \in \Gamma$, and thus induces the same outer action α. Furthermore, the twistings satisfy $f' = fc$, where $c : \Gamma \times \Gamma \to Z(A)^*$ is a 2-coboundary. If Eqs. (2.1) are satisfied, we say that the twisting f (or, alternatively, the corresponding crossed product) *realizes* the outer action α which is induced by β. If we want to emphasize the action and the twisting we denote the corresponding crossed product by $A_\alpha^f * \Gamma$.

Fix $\beta : \Gamma \to \operatorname{Aut}(A)$ and $f : \Gamma \times \Gamma \to A^*$ as above. Then the set of all the twistings that realize the induced outer action α is given by

Proposition 2.1 (*[7,13], See Also [2], Proposition 4.1*).

(1) *If $\beta : \Gamma \to \mathrm{Aut}(A)$ and $f_0 : \Gamma \times \Gamma \to A^*$ satisfy conditions (2.1), then all the twistings that realize the outer action α (induced from β) are of the form $f' = f_0 g'$, where $g' : \Gamma \times \Gamma \to Z(A)^*$ is a 2-cocycle.*
(2) *The crossed products $A * \Gamma$ admitting an outer action $\alpha : \Gamma \to \mathrm{Out}(A)$ are classified by $H^2(\Gamma, Z(A)^*)$.*

Proposition 2.1 allows us to characterize an n-parameter homogeneous deformation of a crossed product $A_\alpha^{f_0} * \Gamma$.

Corollary 2.2. *Let $A((t_1, \dots, t_n))_\alpha^{f_0} * \Gamma = A((t_1, \dots, t_n)) \otimes_A A_\alpha^{f_0} * \Gamma$ be a crossed product obtained by an extension of scalars (f_0 takes its values in A^*). Then $R' = A((t_1, \dots, t_n))_\alpha^{f'} * \Gamma$ is an n-parameter homogeneous deformation of $A_\alpha^{f_0} * \Gamma$ if $f' = f_0 g'$ for some 2-cocycle, $g' : \Gamma \times \Gamma \to U_n$, where $U_n = \{1 + \sum_{i=1}^n t_i a_i \mid a_i \in Z(A[[t_1, \dots, t_n]])\}$ is the subgroup of 1-units of $Z(A[[t_1, \dots, t_n]])^*$.*

2.2

Proposition 1.3, Corollary 1.4, Theorem 1.7 involve p-independence and p-degree. Here are the precise definitions:

Definition 2.3. Let $K_1 \subset K_2$ be an extension of fields of characteristic $p > 0$. A subset $S \subset K_2$ is said to be p-independent over K_1 if $K_1(K_2^p)(T) \neq K_1(K_2^p)(S)$ for any proper subset $T \subset S$. We say that $p\text{-}\deg(K) = r$ if there is a subset $S \subset K$ of cardinality r which is p-independent over K^p such that $K^p(S) = K$. Such S is called a p-basis of K (over K^p).

Let $K_1 \subset K_2$ be an extension of fields. Recall that a derivation of K_2 over K_1 is a K_1-linear map $\partial : K_2 \to K_2$ with $\partial(xy) = x\partial(y) + y\partial(x) \forall x, y \in K_2$.

Given derivations $\partial_1, \dots, \partial_r$ of K_2 over K_1, the Jacobian map with respect to $\partial_1, \dots, \partial_r$ is defined by

$$J = J_{\partial_1, \dots, \partial_r} : (K_2)^r \to K_2$$
$$J(y^{(1)}, \dots, y^{(r)}) = \det[\partial_i(y^{(j)})].$$

The following is a criterion for p-independence:

Theorem 2.4 (*See [17] Section 4.3*). *Let $K_1 \subset K_2$ be an extension of fields, then the elements $y^{(1)}, \dots, y^{(r)} \in K_2$ are p-independent over K_1 if and only if there exist derivations $\partial_1, \dots, \partial_r$ of K_2 over K_1 such that the Jacobian $J(y^{(1)}, \dots, y^{(r)})$ with respect to $\partial_1, \dots, \partial_r$ does not vanish.*

Corollary 2.5. *Let K be a field of characteristic $p > 0$ and let $K((t_1, \dots, t_n))$ be the field of power series on n indeterminates over K. Then $p\text{-}\deg(K((t_1, \dots, t_n))) = p\text{-}\deg(K) + n$.*

Proof. Let $r = p\text{-}\deg(K)$ (if the p-degree is infinite, the corollary follows at once). Let $\{y^{(1)}, \dots, y^{(r)}\}$ be a p-basis of K. Then we claim that $\{y^{(1)}, \dots, y^{(r)}, t_1, \dots, t_n\}$ is a p-basis of $K((t_1, \dots, t_n))$. Clearly, $K^p(y^{(1)}, \dots, y^{(r)}) = K$. Hence $(K((t_1, \dots, t_n)))^p(y^{(1)}, \dots, y^{(r)}, t_1, \dots, t_n) = K((t_1, \dots, t_n))$. Next, let $\partial_1, \dots, \partial_r$ be derivations of K over K^p, such that $J_{\partial_1, \dots, \partial_r}(y^{(1)}, \dots, y^{(r)}) \neq 0$ as provided by Theorem 2.4. Extend these derivations to $K((t_1, \dots, t_n))$ trivially and add the n derivations $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$ of $K((t_1, \dots, t_n))$. Then

$$J_{\partial_1, \dots, \partial_r, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}}(y^{(1)}, \dots, y^{(r)}, t_1, \dots, t_n) = J_{\partial_1, \dots, \partial_r}(y^{(1)}, \dots, y^{(r)}) \neq 0$$

and hence $\{y^{(1)}, \dots, y^{(r)}, t_1, \dots, t_n\}$ are p-independent over $(K((t_1, \dots, t_n)))^p$. □

Lemma 2.6 (*See [3], Lemma 13*). *Let $K_1 \subset K_2$ be a finite separable extension of fields of characteristic $p > 0$. Then $p\text{-}\deg(K_1) = p\text{-}\deg(K_2)$.*

Proof. Let X be a p-basis of K_1 over K_1^p. We claim that X serves also as a p-basis of K_2 over K_2^p. To prove that the set X remains p-independent over K_2^p, extend a base of K_1 over K_1^p to a base of K_2 over K_1^p and then extend the derivations of K_1 whose existence is guaranteed by Theorem 2.4 trivially on the new elements of the base. This extension does not change the original Jacobian.

Next, we need to prove that $K_2^p(X) = K_2$. If this is false, there exists an element $y \in K_2 \setminus K_2^p(X)$. But then the polynomial $P(t) = t^p - y^p$ is irreducible over $K_2^p(X)$ and hence $K_2^p(X) \subset K_2$ is not separable. This contradicts the separability assumption since $K_1 = K_1^p(X) \subset K_2^p(X) \subset K_2$. $\square$

Note that the Jacobian satisfies $J(y^{(1)}, \dots, y_1^{(i)} y_2^{(i)}, \dots, y^{(r)}) = y_1^{(i)} J(y^{(1)}, \dots, y_2^{(i)}, \dots, y^{(r)}) + y_2^{(i)} J(y^{(1)}, \dots, y_1^{(i)}, \dots, y^{(r)})$. Inductively

$$J\left(y^{(1)}, \dots, \prod_{k=1}^{m} y_k^{(i)}, \dots, y^{(r)}\right) = \sum_{k=1}^{m} \left(\prod_{l \neq k} y_l^{(i)}\right) J(y^{(1)}, \dots, y_k^{(i)}, \dots, y^{(r)}). \tag{2.2}$$

Applying (2.2) in each component yields

$$J\left(\prod_{k=1}^{m} y_k^{(1)}, \prod_{k=1}^{m} y_k^{(2)}, \dots, \prod_{k=1}^{m} y_k^{(r)}\right) = \beta \cdot \sum_{1 \le l_1, \dots, l_r \le m} \frac{J(y_{l_1}^{(1)}, y_{l_2}^{(2)}, \dots, y_{l_r}^{(r)})}{\prod_{j=1}^{r} y_{l_j}^{(j)}}, \tag{2.3}$$

where $\beta = \prod_{j,k} y_k^{(j)} (1 \le k \le m, 1 \le j \le r)$.

From (2.3) we obtain the following

Lemma 2.7. *Let $K_1 \subset K_2$ be an extension of fields and let $J = J_{\partial_1, \dots, \partial_r}$ be the Jacobian map with respect to some K_1-linear derivations $\partial_1, \dots, \partial_r$. Let r, m be two (positive) integers such that $r < m$. Then, for any $m \cdot r$ elements $y_k^{(j)}$, $1 \le k \le m$, $1 \le j \le r$ in K_2*

$$\sum_{B \subset \{1,2,\dots,m\}} (-1)^{|B|} \cdot \frac{J\left(\prod_{k \in B} y_k^{(1)}, \prod_{k \in B} y_k^{(2)}, \dots, \prod_{k \in B} y_k^{(r)}\right)}{\prod_{j=1}^{r} \prod_{k \in B} y_k^{(j)}} = 0. \tag{2.4}$$

(Here B runs over all subsets of $\{1, 2, \dots, m\}$.)

Proof. First, we decompose the summation according to (2.3) (until none of the summands contain products inside the Jacobian). Then we compute the coefficient of every summand $\frac{J(y_{l_1}^{(1)}, y_{l_2}^{(2)}, \dots, y_{l_r}^{(r)})}{\prod_{j=1}^{r} y_{l_j}^{(j)}}$, $1 \le l_1, \dots, l_r \le m$. Any $L = \{l_1, \dots, l_r\}$ appears exactly once in every summand of (2.4) that corresponds to a set $B \supseteq L$. Suppose $|L| = l (\le r < m)$, then the number of sets $B \subseteq \{1, 2, \dots, m\}$ of cardinality $s \ge l$ containing L is $\binom{m-l}{s-l}$.

Since $l \le r < m$, the contribution of $\frac{J(y_{l_1}^{(1)}, y_{l_2}^{(2)}, \dots, y_{l_r}^{(r)})}{\prod_{j=1}^{r} y_{l_j}^{(j)}}$ to the summation is

$$\sum_{s=l}^{m} (-1)^s \binom{m-l}{s-l} = (-1)^l \sum_{i=0}^{m-l} (-1)^i \binom{m-l}{i} = 0. \quad \square$$

3. The twisting problem

Our strategy for the proof of Theorem 1.6 is as follows. We first show that if $A_\alpha^{f_0} * \Gamma$ is a crossed product, if A is finitely generated as a module over its center and if conditions (A) and (B) in Corollary 1.4 are satisfied, then there exists a 2-cocycle $g' : \Gamma \times \Gamma \to Z(A)^*$ such that $A_\alpha^{f_0 g'} * \Gamma$ is semisimple.

Next, suppose that $A_\alpha^{f_0} * \Gamma$ satisfies the hypothesis of Theorem 1.6. It is then clear that $A_\alpha^{f_0} * \Gamma$ and therefore $A((t_1, \dots, t_n))_\alpha^{f_0} * \Gamma$ satisfy condition (A) and so if n is large enough condition (B) will be satisfied as well (see Corollary 2.5) and hence by the first step one can find a 2-cocycle $g' : \Gamma \times \Gamma \to Z(A((t_1, \dots, t_n)))^*$ such that $A((t_1, \dots, t_n))_\alpha^{f_0 g'} * \Gamma$ is semisimple.

The last step (done in Section 4) is to show that such g' may be chosen so that it takes its values in the group $U_n \subset Z(A[[t_1, \ldots, t_n]])^*$ of 1-units (see Corollary 2.2).

We start by recalling the twisting problem.

The twisting problem: Let $A_\alpha^{f_0} * \Gamma$ be a given crossed product and assume the necessary conditions (for semisimplicity) (A) and (B) are satisfied. Does there exist a semisimple crossed product $A_\alpha^{f'} * \Gamma$?

The twisting problem was solved in [3,4] in the case when $A = K$ is a (commutative) field.

In [2] there is a solution of the twisting problem in the case when Γ is cyclic and A is finitely generated as a module over its center.

We show that the twisting problem has a positive answer in the case when A is finitely generated as a module over its center.

Theorem 3.1. *Let $A_\alpha^{f_0} * \Gamma$ be a crossed product, where A is a semisimple algebra which is finitely generated as a module over its center. Assume the necessary conditions (for semisimplicity)* (A) *and* (B) *are satisfied. Then there exists a semisimple crossed product $A_\alpha^{f'} * \Gamma$.*

The first and main part of the construction of the twisting f' is in case A is a central simple algebra. This is done in Section 3.1. Then, in Section 3.2, we show how to extend the construction of a semisimple crossed product to the case where A is semisimple (and finitely generated over its center) and the induced action of Γ on the simple components of A is transitive (i.e. the number of orbits is 1). Finally, in Section 3.3, we extend the construction to any number of orbits.

3.1. Central simple base ring

Let A be a central simple algebra and let $A_\alpha^{f_0} * \Gamma$ be a crossed product. We need to show that if the field $Z(A)$ and the group Γ satisfy conditions (A) and (B), then there exists a twisting $f' : \Gamma \times \Gamma \to A^*$ (realizing the outer action α) such that $A_\alpha^{f'} * \Gamma$ is semisimple. In view of Proposition 2.1 we have to find a 2-cocycle $g' : \Gamma \times \Gamma \to Z(A)^*$ such that $A_\alpha^{f_0 g'} * \Gamma$ is semisimple.

Recall (Proposition 1.2) that such a crossed product $A_\alpha^{f_0 g'} * \Gamma$ is semisimple if and only if the twisted group ring $Z(A)^{f_0 g'|_H} H$ is semisimple, where $H = \ker(\alpha) \lhd \Gamma$. Hence, if we denote by $f = f_0|_H : H \times H \to Z(A)^*$ (the restriction of f_0 to the subgroup H), the twisting problem will be solved if we find a 2-cocycle $g : H \times H \to Z(A)^*$ such that $Z(A)^{fg} H$ is semisimple and the class $[g]$ is in the image of $\mathrm{res}_H^\Gamma : H^2(\Gamma, Z(A)^*) \to H^2(H, Z(A)^*)$. It is therefore sufficient to prove the following

Theorem 3.2. *Let Γ be a finite group, K a field of characteristic $p > 0$, and $\eta : \Gamma \to \mathrm{Aut}(K)$ an action of Γ on K with kernel H. Let $[f] \in H^2(H, K^*)$ be any class. Then there is a class $[g] \in \mathrm{Im}(\mathrm{res}_H^\Gamma : H^2(\Gamma, K^*) \to H^2(H, K^*))$ such that $K^{f \cdot g} H$ is semisimple if and only if any p-Sylow subgroup P of H is abelian with a normal complement in H and further the rank of P does not exceed the p-degree of K over K^p.*

Note that the case where $f \equiv 1$ is solved in [4].

Proof of Theorem 3.2. We first assume that H is a p-group and hence abelian. Let $H = \oplus_{1 \le i \le r} \mathbb{Z}/p^{e_i}\mathbb{Z} = \langle x_1 \rangle \oplus \cdots \oplus \langle x_r \rangle$. We have $r = \mathrm{rank}(H) \le p\text{-}\deg(K)$.

We need the following result:

Proposition 3.3 (*See also [3], Proposition 9*). *Let K be a field of characteristic $p > 0$, and Q a finite subgroup of $\mathrm{Aut}(K)$. Let $S \subset K^*$ be a Q-submodule and let M be a $\mathbb{Z}_{p^e} Q$-module. Then*

$$H^2(M, S) \simeq \mathrm{Hom}_{\mathbb{Z}_{p^e}}(M, S/S^{p^e}).$$

Furthermore, taking Q-invariants

$$H^2(M, S)^Q \simeq \mathrm{Hom}_{\mathbb{Z}_{p^e} Q}(M, S/S^{p^e}).$$

Proof. Since the Schur multiplier $H_2 M$ is a p-group and since S has no p-torsion, it follows that $\mathrm{Hom}(H_2 M, S) = 0$. Hence, by the Universal Coefficients Theorem, $H^2(M, S)$ is isomorphic with $\mathrm{Ext}(M, S)$ and therefore with $\mathrm{Hom}_{\mathbb{Z}_{p^e}}(M, S/S^{p^e})$. $\square$

We apply Proposition 3.3 for $M = H$, $e = \max_{1\le i\le r}\{e_i\}$ and $Q = \Gamma/H$. The action of Q on H is determined by the group extension (H is abelian) $1 \to H \to \Gamma \to \Gamma/H = Q \to 1$.

Let $[\gamma] \in H^2(H, K^*)$ and let $\mu_\gamma : H \to K^*/(K^*)^{p^e}$ be the corresponding $\mathbb{Z}_{p^e}$-morphism from Proposition 3.3 (here S is K^* itself). Then μ_γ may be represented by an r-tuple $(w_1^{p^{e-e_1}}(K^*)^{p^e}, w_2^{p^{e-e_2}}(K^*)^{p^e}, \ldots, w_r^{p^{e-e_r}}(K^*)^{p^e})$, where $\mu_\gamma(x_i) = w_i^{p^{e-e_i}}(K^*)^{p^e} \in K^*/(K^*)^{p^e}$. Furthermore, there exist a K-basis $\{u_h\}_{h\in H}$ of the twisted group algebra $K^\gamma H$ such that $u_{x_i}^{p^{e_i}} = w_i \in K^*$. Hence, by Proposition 1.3 $K^\gamma H$ is semisimple if and only if the elements $w_1, w_2, \ldots, w_r$ are p-independent over K^p.

Let $(f_1^{p^{e-e_1}}(K^*)^{p^e}, \ldots, f_r^{p^{e-e_r}}(K^*)^{p^e})$ be the r-tuple which corresponds to our element (of Theorem 3.2) $[f] \in H^2(H, K^*)$. By the preceding paragraphs, solving the twisting problem reduces to finding an r-tuple $(g_1^{p^{e-e_1}}(K^*)^{p^e}, \ldots, g_r^{p^{e-e_r}}(K^*)^{p^e})$, $g_i \in K^*$ such that the corresponding element $[g] \in H^2(H, K^*)$ is in the image of the restriction map $\mathrm{res}_H^\Gamma : H^2(\Gamma, K^*) \to H^2(H, K^*)$ and such that the elements $f_i g_i$, $1 \le i \le r$ are p-independent over K^p. Since the image of res_H^Γ is contained in $H^2(H, K^*)^Q$, it follows by Proposition 3.3 that for any such g, the map $\mu_g : H \to K^*/(K^*)^{p^e}$ given by $\mu_g(x_i) = g_i^{p^{e-e_i}}(K^*)^{p^e}$, $1 \le i \le r$ should be $\mathbb{Z}_{p^e}Q$-linear.

The following result will be useful in finding elements in the image of $\mathrm{res}_H^\Gamma : H^2(\Gamma, K^*) \to H^2(H, K^*)^Q$.

Theorem 3.4 (*See [3], Proof of Theorem 5*). *With the above notation, let $S \subset K^*$ be a Q-submodule such that S/S^p is free over $\mathbb{Z}_p Q_p$, where Q_p is a p-Sylow subgroup of Q. Then the restriction map $H^2(\Gamma, S) \xrightarrow{\mathrm{res}} H^2(H, S)^Q$ is surjective.*

Proof. Since S/S^p is free over $\mathbb{Z}_p Q_p$ and since S has no p-torsion we obtain that for every $q > 0$, $H^q(Q_p, S) = 0$ (see [16], page 143, Thm. 6). Next, the composition $H^q(Q, S) \xrightarrow{\mathrm{res}} H^q(Q_p, S) \xrightarrow{\mathrm{cor}} H^q(Q, S)$ is multiplication by $|Q : Q_p|$, which is coprime to p, and hence $(H^q(Q, S))_p \xrightarrow{\mathrm{res}} H^q(Q_p, S)$ is injective, where $(H^q(Q, S))_p$ is the p-part of $H^q(Q, S)$. It follows that $(H^q(Q, S))_p = 0$ for every $q > 0$. Now, since H is an abelian p-group and S has no p-torsion we obtain $H^1(H, S) = 0$. Then the LHS spectral sequence yields the exact sequence (see [15], page 305, 11.3)

$$H^2(\Gamma, S) \xrightarrow{\mathrm{res}} H^2(H, S)^Q \xrightarrow{d} H^3(Q, S).$$

Since $H^2(H, S)^Q$ is a p-group, the image of d is in $(H^3(Q, S))_p$ and we are done. □

We wish to find appropriate elements $g_1, \ldots, g_r$ in a Q-submodule $S \subset K^*$ such that S/S^p is free over Q. Suppose that such a submodule S exists and denote its embedding in K^* by ι. Consider the commutative diagram

$$\begin{array}{ccc} H^2(\Gamma, K^*) & \xrightarrow{\mathrm{res}} & H^2(H, K^*)^Q \\ \iota_* \uparrow & & \iota_* \uparrow \\ H^2(\Gamma, S) & \xrightarrow{\mathrm{res}} & H^2(H, S)^Q \end{array} \quad . \tag{3.1}$$

The element $[g] = (g_1^{p^{e-e_1}}(K^*)^{p^e}, \ldots, g_r^{p^{e-e_r}}(K^*)^{p^e}) \in H^2(H, K^*)^Q$ is the image under ι_* of $(g_1^{p^{e-e_1}}S^{p^e}, \ldots, g_r^{p^{e-e_r}}S^{p^e}) \in H^2(H, S)^Q$, and by Theorem 3.4 comes from $H^2(\Gamma, S)$. Hence $[g]$ is restricted from $H^2(\Gamma, K^*)$. And indeed, such a submodule $S \subset K^*$ was constructed in [4]. Before we recall the construction we make the following definition: an element $z \in K$ is called *Q-normal* if $\sigma(z) \ne z$ for every $1 \ne \sigma \in Q$.

Proposition 3.5 (*[4], Propositions 1, 2*). *Let p-$\deg(K) \ge r$. Then for all but finitely many Q-normal elements $z \in K$ there exist p-independent elements (over K^p) $a_1, \ldots, a_r$ in K^Q satisfying*

(1) *Let S be the Q-submodule of K^* generated by the elements $\{1 + a_k z^p\}_{1\le k\le r}$, then $F = S/S^{p^e}$ is freely generated by $(1 + a_k z^p)S^{p^e}$ over $\mathbb{Z}_{p^e}Q$.*

(2) *The elements $b_j = \prod_{k=1}^r \prod_{\sigma\in Q}(1 + a_k\sigma(z)^p)^{(\sigma^{-1}(x_j))_k} \in S$, $1 \le j \le r$ are p-independent over K^p, where $(h)_k$ is defined by $h = \prod_{k=1}^r x_k^{(h)_k} \in H$.*

Furthermore, for any choice of r elements $\{c_1, \ldots, c_r\} \subset K^Q$ which are p-independent over K^p there are elements $\{u_1, \ldots, u_r\} \subset (K^Q)^p$ such that $a_i = c_i u_i$, and once $u_1, \ldots, u_{i-1}$ have been chosen, all but a finite number of elements of $(K^Q)^p$ qualify as u_i.

We shall assume for the rest of this section that $r = \operatorname{rank}(H) > 0$ (the case $r = 0$ means that H is a p'-group and Theorem 3.2 is obvious). Under this assumption K is not perfect and hence infinite. Therefore, elements $z, a_1, \ldots, a_r \in K$ satisfying Proposition 3.5 do exist. Taking such $r + 1$ elements, we let

$$y_k^{(j)} = \prod_{\sigma \in Q} (1 + a_k \sigma(z)^p)^{(\sigma^{-1}(x_j))_k}, \quad 1 \le k, j \le r. \tag{3.2}$$

Note that the elements b_j in Proposition 3.5(2) may be written as $b_j = \prod_{k=1}^r y_k^{(j)}$, $1 \le j \le r$.

Proposition 3.6. *For any $1 \le k \le r$ the map $\mu_k : H \to S/S^{p^e}$ given by $\mu_k(x_j) = y_k^{(j)} S^{p^e}$, $1 \le j \le r$ is $\mathbb{Z}_{p^e} Q$-linear. Moreover, under the isomorphism in Proposition 3.3, μ_k corresponds to a cohomology class in the image of the restriction map $\operatorname{res}_H^\Gamma : H^2(\Gamma, S) \to H^2(H, S)^Q$.*

Proof. Let $g \in Q$. For every $1 \le j \le r$, let $g(x_j) = \prod_{i=1}^r x_i^{n_{i,j}}$, where $n_{i,j} = n_{i,j}(g)$. Then

$$\begin{aligned} \mu_k(g(x_j)) &= \prod_{i=1}^r \mu_k(x_i)^{n_{i,j}} = \prod_{i=1}^r \prod_{\sigma \in Q} (1 + a_k \sigma(z)^p)^{(\sigma^{-1}(x_i))_k n_{i,j}} \\ &= \prod_{\sigma \in Q} (1 + a_k \sigma(z)^p)^{\sum_{i=1}^r (\sigma^{-1}(x_i)^{n_{i,j}})_k}. \end{aligned} \tag{3.3}$$

Note that $(\sigma^{-1}(g(x_j)))_k = (\sigma^{-1}(\prod_{i=1}^r x_i^{n_{i,j}}))_k = (\prod_{i=1}^r (\sigma^{-1}(x_i^{n_{i,j}})))_k = \sum_{i=1}^r (\sigma^{-1}(x_i)^{n_{i,j}})_k$. Hence (3.3) becomes

$$\mu_k(g(x_j)) = \prod_{\sigma \in Q} (1 + a_k \sigma(z)^p)^{(\sigma^{-1}(g(x_j)))_k} = \prod_{\tau \in Q} (1 + a_k g\tau(z)^p)^{(\tau^{-1}(x_j))_k} = g(\mu_k(x_j)). \tag{3.4}$$

Hence μ_k is $\mathbb{Z}_{p^e} Q$-linear.

For the second part we use Theorem 3.4. We need to show that S/S^p is free over $\mathbb{Z}_p Q_p$, where Q_p is a p-Sylow subgroup of Q. Indeed, this requirement is valid since, by Proposition 3.5, $F = S/S^{p^e}$ is free over $\mathbb{Z}_{p^e} Q$. $\square$

Corollary 3.7. *With the above notation, the element $(b_1^{p^{e-e_1}} S^{p^e}, \ldots, b_r^{p^{e-e_r}} S^{p^e}) \in H^2(H, S)$ is in the image of the restriction map $\operatorname{res}_H^\Gamma : H^2(\Gamma, S) \to H^2(H, S)^Q$.*

Proof. By Proposition 3.6, for every $1 \le k \le r$ there exists $[y_k'] \in H^2(\Gamma, S)$ such that $\operatorname{res}_H^\Gamma([y_k']) = (y_k^{(1)p^{e-e_1}} S^{p^e}, \ldots, y_k^{(r)p^{e-e_r}} S^{p^e})$. Thus, $\operatorname{res}_H^\Gamma(\sum_{k=1}^r [y_k']) = (b_1^{p^{e-e_1}} S^{p^e}, \ldots, b_r^{p^{e-e_r}} S^{p^e})$. $\square$

Observe that the element $[\hat{g}] = (b_1^{p^{e-e_1}} (K^*)^{p^e}, \ldots, b_r^{p^{e-e_r}} (K^*)^{p^e})$ solves the twisting problem in the case when $f \equiv 1$. This follows from the construction in Proposition 3.5 that provides the p-independence of the elements $\{b_j\}_{j=1}^r$, and from Corollary 3.7, which says that $[\hat{g}]$ is in fact an image under $\operatorname{res}_H^\Gamma : H^2(\Gamma, K^*) \to H^2(H, K^*)^Q$.

In the general case (when f is not necessarily 1) we will need a variation of $[\hat{g}]$. The class $[\hat{g}]$ is decomposed into the r classes determined by the products $\prod_{k=1}^r y_k^{(j)} (= b_j)$. Then we "glue" these classes again in a suitable way (based on the given f).

Proposition 3.8. *For any $[f] = (f_1^{p^{e-e_1}} (K^*)^{p^e}, \ldots, f_r^{p^{e-e_r}} (K^*)^{p^e}) \in H^2(H, K^*)$ there exists $B(f) \subset \{1, 2, \ldots, r\}$ such that the elements $f_1 \cdot \prod_{k \in B(f)} y_k^{(1)}, \ldots, f_r \cdot \prod_{k \in B(f)} y_k^{(r)}$ are p-independent.*

Proof. By Proposition 3.5(2), the elements $b_1, \ldots, b_r$ are p-independent over K^p. By Theorem 2.4, there exist K^p-linear derivations $\partial_1, \ldots, \partial_r : K \to K$ such that

$$J(b_1, b_2, \ldots, b_r) = J\left(\prod_{k=1}^r y_k^{(1)}, \prod_{k=1}^r y_k^{(2)}, \ldots, \prod_{k=1}^r y_k^{(r)}\right) \neq 0, \tag{3.5}$$

where $J = J_{\partial_1, \ldots, \partial_r}$. For convenience, we denote

$$y_{r+1}^{(j)} = f_j, \quad 1 \le j \le r.$$

Plugging $y_k^{(j)}$, $1 \le k \le m = r+1$, $1 \le j \le r$ in (2.4), we have

$$\sum_{B\subset\{1,\ldots,r+1\}} (-1)^{|B|} \cdot \frac{J\left(\prod_{k\in B} y_k^{(1)}, \prod_{k\in B} y_k^{(2)}, \ldots, \prod_{k\in B} y_k^{(r)}\right)}{\prod_{j=1}^{r}\prod_{k\in B} y_k^{(j)}} = 0. \tag{3.6}$$

We decompose this sum into two summands $\Sigma_1 + \Sigma_2 = 0$, where

$$\Sigma_1 = \sum_{r+1\notin B} (-1)^{|B|} \cdot \frac{J\left(\prod_{k\in B} y_k^{(1)}, \prod_{k\in B} y_k^{(2)}, \ldots, \prod_{k\in B} y_k^{(r)}\right)}{\prod_{j=1}^{r}\prod_{k\in B} y_k^{(j)}}.$$

$$\Sigma_2 = \sum_{r+1\in B} (-1)^{|B|} \cdot \frac{J\left(\prod_{k\in B} y_k^{(1)}, \prod_{k\in B} y_k^{(2)}, \ldots, \prod_{k\in B} y_k^{(r)}\right)}{\prod_{j=1}^{r}\prod_{k\in B} y_k^{(j)}}.$$

Clearly, if $\Sigma_2 \neq 0$, then for some $B \subset \{1, 2, \ldots, r+1\}$ with $r+1 \in B$

$$J\left(\prod_{k\in B} y_k^{(1)}, \prod_{k\in B} y_k^{(2)}, \ldots, \prod_{k\in B} y_k^{(r)}\right) \neq 0.$$

This would imply for $B(f) = B \setminus \{r+1\}$ that the elements $f_1 \cdot \prod_{k\in B(f)} y_k^{(1)}, \ldots, f_r \cdot \prod_{k\in B(f)} y_k^{(r)}$ are p-independent over K^p as desired.

In order to prove that Σ_2 does not vanish, we show that $\Sigma_1 \neq 0$. By (3.5) we have $J(b_1, b_2, \ldots, b_r) \neq 0$ and hence the term in Σ_1 which corresponds to $B_0 = \{1, 2, \ldots, r\}$ is nonzero. To prove that $\Sigma_1 \neq 0$ we show that all other terms in Σ_1 (i.e. terms that correspond to proper subsets of $\{1, 2, \ldots, r\}$) vanish. Indeed, by the definition of $y_k^{(j)}$ in Proposition 3.5, we have that $y_k^{(j)} \in K^p(a_k)$ for all $1 \le j, k \le r$. It follows that $\{\prod_{k\in B} y_k^{(1)}, \prod_{k\in B} y_k^{(2)}, \ldots, \prod_{k\in B} y_k^{(r)}\} \subset K^p(\{a_k\}_{k\in B})$. Now, if $B \subsetneq \{1, 2, \ldots, r\}$ then $p\text{-}\deg(K^p(\{a_k\}_{k\in B})) < r$ and hence the elements $\prod_{k\in B} y_k^{(1)}, \prod_{k\in B} y_k^{(2)}, \ldots, \prod_{k\in B} y_k^{(r)}$ are p-dependent over K^p. By Theorem 2.4 we obtain

$$J\left(\prod_{k\in B} y_k^{(1)}, \prod_{k\in B} y_k^{(2)}, \ldots, \prod_{k\in B} y_k^{(r)}\right) = 0, \quad B \subsetneq \{1, 2, \ldots, r\}. \quad \square \tag{3.7}$$

We now return to the hypothesis of Theorem 3.2. Let $[f] = (f_1^{p^{e-e_1}}(K^*)^{p^e}, \ldots, f_r^{p^{e-e_r}}(K^*)^{p^e}) \in H^2(H, K^*)$ be any class. Then, under the conditions of the theorem, Proposition 3.8 says that there exists a subset $B = B(f) \subset \{1, 2, \ldots, r\}$ such that $A^{f\cdot\iota_* g_B} * H$ is semisimple, where

$$[g_B] = \left(\prod_{k\in B}(y_k^{(1)})^{p^{e-e_1}}(S)^{p^e}, \ldots, \prod_{k\in B}(y_k^{(r)})^{p^{e-e_r}}(S)^{p^e}\right) \in H^2(H, S). \tag{3.8}$$

By Proposition 3.6, $[g_B]$ is in the image of $H^2(\Gamma, S) \xrightarrow{\text{res}} H^2(H, S)$, and by the commutative diagram (3.1), $[\iota_* g_B] = \iota_*[g_B]$ is in the image of $H^2(\Gamma, K^*) \xrightarrow{\text{res}} H^2(H, K^*)$. This completes the proof of Theorem 3.2 in the case when H is a p-group.

We now drop the assumption of H being a p-group. Let $P < H$ be a p-Sylow subgroup of H and let $N \lhd H$ be its normal complement in H. Then $N = \{h \in H : p \nmid o(h)\}$ and hence is normal in Γ. Put $\bar{\Gamma} = \Gamma/N$ and $\bar{H} = H/N$. The above conditions yield a natural isomorphism $\Psi : P \to \bar{H}$ which induces an isomorphism $\Psi_* : H^2(\bar{H}, K^*) \to H^2(P, K^*)$.

Let $\bar{\tau} : \bar{\Gamma} \to \mathrm{Aut}(K)$ be the induced action with $\ker(\bar{\tau}) = \bar{H}$ and let $[\bar{f}] = \Psi_*^{-1}(\mathrm{res}_P^H([f])) \in H^2(\bar{H}, K^*)$, where $[f]$ is the given element in $H^2(H, K^*)$. Since $\bar{H}$ is a p-group, there exists $[\bar{g}] \in H^2(\bar{\Gamma}, K^*)$ such that $K^{(\mathrm{res}_{\bar{H}}^{\bar{\Gamma}}\bar{g})\cdot\bar{f}}\bar{H}$ is semisimple. Letting $[g] = \mathrm{res}^{\Gamma}_{H}(\inf^{\Gamma}_{\bar{\Gamma}}(\bar{g})) \in H^2(H, K^*)$, we have $\mathrm{res}^{H}_{P}([g \cdot f]) = \Psi_*((\mathrm{res}^{\bar{\Gamma}}_{\bar{H}}[\bar{g}]) \cdot [\bar{f}])$, and therefore $K^{\mathrm{res}^{H}_{P}(g\cdot f)}P \simeq K^{(\mathrm{res}^{\bar{\Gamma}}_{\bar{H}}\bar{g})\cdot \bar{f}}\bar{H}$, which is semisimple. By Proposition 1.3 (see also [2] Theorem 3.3, 5 $\Leftarrow$ 6) $K^{g\cdot f}H$ is semisimple. $\square$

3.2. Transitive action on the simple components

Let $A = A_1 \oplus \cdots \oplus A_s$ be semisimple and finitely generated over its center $Z(A) = K_1 \oplus \cdots \oplus K_s$. In this step we assume that the induced action of Γ on the simple components of A is transitive. That is for every $i = 1, \ldots, s$ there is an element $\sigma_i \in \Gamma$ such that $\sigma_i(A_1) = A_i$. We assume of course that a crossed product $A^{f_0}_{\alpha} * \Gamma$ is given.

Let $\Gamma_1 = \mathrm{stab}_{\Gamma}(A_1)$. Restricting $f_0|_{\Gamma_1\times\Gamma_1} : \Gamma_1 \times \Gamma_1 \to A^* = \prod_{i=1}^{s} A^*_i$ and projecting onto the first component we obtain the twisting $\hat{f}_0 = \mathrm{proj} \circ \mathrm{res}^{\Gamma}_{\Gamma_1}(f_0) : \Gamma_1 \times \Gamma_1 \to A^*_1$. By the preceding step there is a 2-cocycle $\hat{g} : \Gamma_1 \times \Gamma_1 \to K^*_1$ such that the crossed product $A_1{}^{\hat{f}_0\hat{g}}_{\alpha|\Gamma_1} * \Gamma_1$ is semisimple. Now, by Shapiro's Lemma (see [5], page 73, Prop. 6.2) the map $\mathrm{proj} \circ \mathrm{res}^{\Gamma}_{\Gamma_1}$ induces an isomorphism

$$H^2\left(\Gamma, \prod_{i=1}^{s} K^*_i\right) \to H^2(\Gamma_1, K^*_1).$$

The inverse is $\mathrm{cor}^{\Gamma}_{\Gamma_1} \circ i$, which is the composite of the embedding of K^*_1 into $\prod_{i=1}^{s} K^*_i$ and the correstriction map.

We wish to show that the crossed product $A^{f'}_{\alpha} * \Gamma$ twisted by $f' = f_0(\mathrm{cor} \circ i(\hat{g})) : \Gamma \times \Gamma \to A^*$ is semisimple. Indeed, restricting to Γ_1 and projecting the values to A_1 gives $\mathrm{proj} \circ \mathrm{res}^{\Gamma}_{\Gamma_1}(f') = \mathrm{proj} \circ \mathrm{res}^{\Gamma}_{\Gamma_1}(f_0(\mathrm{cor} \circ i(\hat{g}))) = \mathrm{proj} \circ \mathrm{res}^{\Gamma}_{\Gamma_1}(f_0) \cdot \hat{g} = \hat{f}_0\hat{g}$.

By Proposition 1.1, since the crossed product $A_1{}^{\hat{f}_0\hat{g}}_{\alpha|\Gamma_1} * \Gamma_1$ is semisimple, so is the crossed product $A^{f'}_{\alpha} * \Gamma$.

3.3. General action

Assume now that the induced action of Γ on the simple components of A determines $l \geq 1$ orbits. Let A_j be the direct sum of the simple components in the jth orbit and let f_j, $j = 1, \ldots, l$ be the twisting such that $A_j{}^{f_j}_{\alpha} * \Gamma$ is semisimple. Then the algebra $A^{\prod_{j=1}^{l} f_j}_{\alpha} * \Gamma$ is semisimple. This completes the proof of Theorem 3.1. $\square$

We end this section with the following counterexample which shows that the condition in Theorem 3.1 on the finite generation of A over its center cannot be omitted.

Let $D = \mathbb{F}_2(t)^{t^2}(X, Y)$ be the skew field of fractions generated by X and Y over $\mathbb{F}_2(t)$, defined by the relation $XY = t^2YX$. Note that D is infinite dimensional over its center $\mathbb{F}_2(t)$. We define an automorphism $\bar{\sigma}$ of D by $X^{\bar{\sigma}} = X, Y^{\bar{\sigma}} = tY$. Since $\bar{\sigma}^2$ acts as conjugation by X, we obtain an outer action α of $C_2 = \langle\bar{\sigma}\rangle$ on D. Let $\tilde{f} : C_2 \times C_2 \to D^*$ be the following twisting: $\tilde{f}(1, 1) = \tilde{f}(1, \bar{\sigma}) = \tilde{f}(\bar{\sigma}, 1) = 1$, $\tilde{f}(\bar{\sigma}, \bar{\sigma}) = X$ (see Example 4.2 in [2]). Now, consider the outer action of the quaternion group $Q_8 = \langle \sigma, \tau : \sigma^4 = e, \tau\sigma\tau^{-1} = \sigma^{-1}, \sigma^2 = \tau^2\rangle$ on D via its quotient $C_2 = \langle\bar{\sigma}\rangle = Q_8/\langle\tau\rangle$. One can inflate the twisting $\tilde{f}$ to a twisting $f_0 : Q_8 \times Q_8 \to D^*$. Note that f_0 vanishes on the kernel $\langle\tau\rangle$ of the outer action. By Proposition 2.1 every twisting on Q_8 is of the form f_0g for some $[g] \in H^2(Q_8, \mathbb{F}_2(t)^*)$. We claim that any such element $[g] \in H^2(Q_8, \mathbb{F}_2(t)^*)$ vanishes on the commutator $\langle\tau^2\rangle$ and hence any twisting f_0g is trivial on $\langle\tau^2\rangle$. Indeed, since the Schur multiplier $M(Q_8)$ is trivial, $\mathrm{Hom}(M(Q_8), \mathbb{F}_2(t)^*) = 0$. Hence, by the Universal Coefficient Theorem (see [11] Theorem 15.1, p. 222), any element in $H^2(Q_8, \mathbb{F}_2(t)^*)$ is inflated from $H^2((Q_8)_{Ab}, \mathbb{F}_2(t)^*)$ and thus trivial on the commutator.

It follows that any crossed product $D^{f_0g}_{\alpha} * Q_8$ contains the group ring DC_2 with respect to the subgroup $C_2 = \langle\tau^2\rangle$ and therefore is not semisimple. Nevertheless, the conditions for the twisting problem are satisfied in this case: the kernel of the outer action is $\langle\tau\rangle$ of rank one, which is exactly the p-degree of $\mathbb{F}_2(t)$, the center of D.

4. Homogeneous deformations

In this section we prove Theorems 1.6 and 1.7. We shall do it for A central simple. The proof of the general case, namely when A is semisimple finitely generated over its center, goes along the general lines of Sections 3.2 and 3.3.

Recall from the beginning of Section 3 that, in order to complete the proof of Theorem 1.6, we add sufficiently many indeterminates $t_1, \dots, t_n$ so as to satisfy condition (B) in Corollary 1.4. The last step is to show that a 2-cocycle $g' : \Gamma \times \Gamma \to Z(A((t_1, \dots, t_n)))^*$, such that $A((t_1, \dots, t_n))_\alpha^{f_0 g'} * \Gamma$ is semisimple, may be chosen so that it takes its values in the group $U_n \subset Z(A[[t_1, \dots, t_n]])^*$ of 1-units (see Corollary 2.2). Theorem 1.7 (for central simple algebras) will be proved if we show in addition that n can be taken as $\lambda = \max\{1, r - p\text{-}\deg(Z(A))\}$ (by Section 3 it is clear that any smaller number cannot be suitable since condition (B) in Corollary 1.4 is not satisfied).

We first deal with the 2-cocycle $g : H \times H \to Z(A((t_1, \dots, t_n)))^*$ that lifts to g'. Since $g = g_B$ is constructed by products of the elements $y_k^{(j)}$ (see Eq. (3.8)), we need the following

Proposition 4.1. *Let $K = Z(A((t_1, \dots, t_\lambda)))$, where $\lambda = \max\{1, r - p\text{-}\deg(Z(A))\}$. Then the Q-submodule $S \subset K^*$ in Proposition 3.5 can be chosen so that $S \subset U_\lambda \subset Z(A[[t_1, \dots, t_\lambda]])^*$.*

Proof. Recall from Proposition 3.5 that the submodule $S \subset K^*$ is generated by elements $d_k = 1 + a_k z^p$, $1 \le k \le r$, where $z \in K$ and $a_1, \dots, a_r \in K^Q$. We need to show that $z, a_1, \dots, a_r$ may be chosen so that the d_k's are 1-units.

Take $c_1, \dots, c_r$ as follows. For $1 \le i \le \lambda$ let $c_i = t_i$. For $\lambda < i \le r$ let c_i be any p-independent set in $Z(A)^Q$. This can be done since $r - \lambda \le p\text{-}\deg(Z(A)) = p\text{-}\deg(Z(A)^Q)$, where the latter equality follows from Lemma 2.6.

Next, we have to find an appropriate Q-normal element $z \in K$.

In the case when $Z(A)$ is infinite, there are infinitely many Q-normal elements $z \in Z(A)$, thus there exists at least one that satisfies Proposition 3.5.

If $Z(A)$ is finite, it might happen that no $z \in Z(A)$ qualifies. In this case take any normal basis $\{\sigma(z_0)\}_{\sigma \in Q}$ of $Z(A)$ over $Z(A)^Q$. The element z can now be chosen from the infinite set $\{z_0 t_1^i\}_{i \ge 0}$.

Finally, we may demand that for each $1 \le i \le r$, u_i can be chosen from the infinite subset $\{t_1^{jp}\}_{j \ge 1} \subset (Z(A)^Q)^p$.

The choices of $c_1, \dots, c_r, z, u_1, \dots, u_r$ above guarantee that for all $1 \le k \le r$ the elements $1 + a_k \sigma(z)^p \in 1 + t_1 Z(A[[t_1, \dots, t_\lambda]])^* \subset U_\lambda$. This completes the proof of Proposition 4.1. $\square$

From Proposition 4.1 we obtain that the elements $y_k^{(j)} = \prod_{\sigma \in Q} (1 + a_k \sigma(z)^p)^{(\sigma^{-1}(x_j))_k}$, $1 \le k, j \le r$ given in Eq. (3.2) can be chosen to be 1-units. It follows that the 2-cocycle $g = g_B$ which was constructed in Eq. (3.8) can take its values in $S \subset U_\lambda$.

From Theorem 3.4 we obtain that g is restricted from a 2-cocycle $g' : \Gamma \times \Gamma \to S$ such that $A((t_1, \dots, t_\lambda))_\alpha^{f_0 g'} * \Gamma$ is semisimple. Since $S \subset U_\lambda$, this is the required homogeneous deformation of the crossed product $A_\alpha^{f_0} * \Gamma$.

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