MATH 2433-006
Exam I
ANSWERS

INSTRUCTIONS

Show your work on all problems.

1. Determine if the following sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{(-1)^{n-1}n}{n^2+1}$$

Answer:

$$\frac{(-1)^{n-1}n}{n^2+1} = \frac{(-1)^{n-1}}{n+1/n} \to 0 \text{ as } n \to \infty$$

$\{a_n\}$ *converges* to zero.

2. Determine if the following series converges or diverges. If it converges, find the sum.

$$\sum_{n=1}^{\infty} \sqrt[n]{2}$$

Answer: Since $\sqrt[n]{2} > 1$ for all n, the partial sum $s_n = 2 + \sqrt[2]{2} + \sqrt[3]{2} + \dots \sqrt[n]{2} > n$.

So

$$\lim_{n\to\infty} s_n = \infty$$

The series *diverges*.

3. Determine if the following series converges or diverges. If it converges, find the sum.

$$\sum_{n=1}^{\infty} 6(0.9)^{n-1}$$

Answer: A geometric series with $\mid r \mid < 1$ converges to $1/(1-r)$. So the series *converges* to

$$6(\frac{1}{1-0.9}) = 60.$$

4. Use the integral test to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} ne^{-n}$$

Answer: Integrate $\int xe^{-x}dx$ by parts with $u = x$ and $dv = e^{-x}$ so $du = dx$ and

$v = -e^{-x}$. Then $\int udv = uv - \int vdu$ means

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C$$

So

$$\int_1^\infty xe^{-x}dx = \lim_{t\to\infty} \int_1^t xe^{-x}dx = \lim_{t\to\infty} (-te^{-t} - e^{-t} + 1e^{-1} + e^{-1}) = \frac{2}{e}$$

Since the integral is finite, the series *converges*

5. Use the integral test to determine if the following series converges or diverges.

$$\sum_{n=1}^\infty \frac{1}{\sqrt[5]{n}}$$

Answer: $1/\sqrt[5]{n} = n^{-1/5}$

$$\int x^{-\frac{1}{5}}dx = \frac{5}{4}x^{\frac{4}{5}} + C$$

$$\int_1^\infty x^{-\frac{1}{5}}dx = \lim_{t\to\infty} \int_1^t x^{-\frac{1}{5}}dx = \lim_{t\to\infty} (\frac{5}{4}t^{\frac{4}{5}} - \frac{5}{4}) = \infty$$

Since the integral is infinite, the series *diverges*

6. Determine if the following series converges or diverges.

$$\sum_{n=1}^\infty \frac{n+1}{n\sqrt{n}}$$

Answer: $n\sqrt{n} = n^{3/2}$

$$a_n \frac{n+1}{n\sqrt{n}} = \frac{n+1}{n^{\frac{3}{2}}} > \frac{n}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}} = b_n$$

Since the p series $\sum 1/n^p$ diverges when $p < 1$, the series $\sum b_n$ diverges, so by the comparison test the series $\sum a_n$ *diverges*

7. Determine if the following series converges or diverges.

$$\sum_{n=1}^\infty \frac{\cos^2 n}{n^2+1}$$

Answer: $\cos^2 n \leq 1$, so

$$a_n = \frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2} = b_n$$

Since the p series $\sum 1/n^p$ converges when $p > 1$, the series $\sum b_n$ converges, so by the comparison test the series $\sum a_n$ *converges*

8. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}(-1)^n \sin(\frac{\pi}{n})$$

Answer: let

$$f(x) = \sin(\frac{\pi}{x})$$

Then

$$f'(x) = \cos(\frac{\pi}{x})(\frac{-\pi}{x^2}) < 0 \text{ for } x \geq 1$$

So $f(x)$ is decreasing and $\lim_{x\to\infty} f(x) = \sin(0) = 0$. Thus the series satisfies the hypotheses of the alternating series test, and *converges*

9. Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$$

Answer: Since

$$\frac{1}{\sqrt[4]{n}} \text{ is a decreasing sequence with } \lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$$

the series converges by the alternating series test. Since

$$\sum_{n=1}^{\infty} \mid \frac{(-1)^{n+1}}{\sqrt[4]{n}} \mid = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{4}}}$$

which is a p series with p= $1/4 < 1$, the series of absolute values diverges. So the series is *conditionally convergent*

10. Find the radius of convergence and the interval of convergence the following series.

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

Answer: Consider the ratio test ratio:

$$\mid \frac{x^{n+1}}{\sqrt{n+1}} / \frac{x^n}{\sqrt{n}} \mid = \mid x \mid \sqrt{\frac{n}{n+1}}$$

Since

$$\lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1$$

the ratio test ratio has limit $\mid x \mid$. So the series converges for $\mid x \mid < 1$ and diverges for $\mid x \mid > 1$

Thus the *Radius of Convergence* is 1.

For $x = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

is a p series with $p = 1/2 < 1$ so diverges.

For $x = -1$, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by the alternating series test.

Thus the *Interval of Convergence* is $[-1, 1)$