MATH 2433-006
Exam II
ANSWERS

INSTRUCTIONS

Show your work on all problems.

1. Find a power series representation for the following function. [Use the geometric series.]

$$f(x) = \frac{2}{3-x}$$

Answer: The geometric series is

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \text{ for } \mid r \mid < 1$$

So

$$\frac{2}{3-x} = \frac{2}{3}(\frac{1}{1-(x/3)}) = \frac{2}{3}\sum_{n=0}^{\infty}(\frac{x}{3})^n = \sum_{n=0}^{\infty}\frac{2}{3^{n+1}}x^n$$

2. Find the Taylor series for $f(x)$ centered at the given value of a.

$$f(x) = e^x, \ a = 3$$

Answer: The Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since for $f(x) = e^x$, $f^{(n)}(x) = e^x$ for all n, the Taylor series is

$$\sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

3. Evaluate the following indefinite integral as an infinite series

$$\int x\cos(x^3)dx$$

using

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Answer: From the given series for the cosine,

$$x\cos(x^3) = \sum_{n=0}^{\infty} x(-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!}$$

Thus

$$\int x\cos(x^3)dx = \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{6n+1}}{(2n)!} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$$

4. Approximate f by a Taylor polynomial with degree n at the number a:

$$f(x) = x^{2/3}, \ a = 1, \ n = 3$$

Answer: $f(x)$ and its first three derivatives are

$$\begin{aligned}
f^{(0)}(x) &= x^{2/3} \\
f^{(1)}(x) &= (2/3)x^{-1/3} \\
f^{(2)}(x) &= (-1/3)(2/3)x^{-4/3} \\
f^{(3)}(x) &= (-4/3)(-1/3)(2/3)x^{-7/3}
\end{aligned} \tag{1}$$

so

$$\begin{aligned}
f^{(0)}(1) &= 1 \\
f^{(1)}(1) &= 2/3 \\
f^{(2)}(1) &= -2/9 \\
f^{(3)}(1) &= 8/27
\end{aligned}$$

thus

$$\begin{aligned}
T_3(x) &= 1 + (2/3)\frac{1}{1!}(x-1) + (-2/9)\frac{1}{2!}(x-1)^2 + (8/27)\frac{1}{3!}(x-1)^3 \\
&= 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3
\end{aligned}$$

5. Eliminate the parameter to find a Cartesian equation of the curve

$$x = e^{2t}, \ y = t + 1$$

Answer: Since $e^{2t} = x$, $2t = \ln x$ so $t = \frac{1}{2}\ln x$ and so $y = \frac{1}{2}\ln x + 1$

6. Find an equation of the tangent line to the curve at the point corresponding to the given value of the parameter

$$x = t^4 + 1,\ y = t^3 + t,\ t = -1$$

Answer: when $t = -1$, $x = 2$ and $y = -2$. Also

$$\frac{dx}{dt} = 4t^3$$
$$\frac{dy}{dt} = 3t^2 + 1$$

so for $t = -1$

$$\frac{dx}{dt} = -4$$
$$\frac{dy}{dt} = 4$$

So the equation of the tangent line is $4(x-2) - (-4)(y - -2) = 0$, or $y = -x$

7. Find the exact length of the curve

$$x = 1 + 3t^2,\ y = 4 + 2t^3,\ 0 \le t \le 1$$

Answer: The length is given by the integral

$$\int_0^1 \sqrt{(\frac{dx}{dt})^2 + \frac{dy}{dt})^2} dt$$

Since here

$$\frac{dx}{dt} = 6t$$
$$\frac{dy}{dt} = 6t^2$$

the integral is

$$\int_0^1 \sqrt{36t^2 + 36t^4} dt = 6 \int_0^1 \sqrt{1+t^2} t dt$$

If $u = 1 + t^2$ so $tdt = \frac{1}{2} du$ then

$$\int \sqrt{1+t^2} t dt = \int u^{1/2} \frac{1}{2} du = \frac{1}{3} u^{3/2} + C$$

so

$$6\int_0^1 \sqrt{1+t^2}t dt = 6\frac{1}{3}(1+t^2)^{3/2}|_0^1 = 2(2^{3/2}-1)$$

8. Set up an integral that represents the exact area of the surface obtained by rotating the given curve about the x-axis

$$x = t^3, \; y = t^2, \; 0 \le t \le 1$$

Answer: The area is given by the integral formula

$$\int_a^b 2\pi y\sqrt{(\frac{dx}{dt})^2 + \frac{dy}{dt})^2} dt$$

Here

$$\frac{dx}{dt} = 3t^2$$
$$\frac{dy}{dt} = 2t$$

so the formula becomes

$$\int_0^1 2\pi t^2\sqrt{9t^4+4t^2} dt$$

9. Find the slope of the tangent line to the given polar curve at the point specified by the value of θ

$$r = 2\sin\theta, \; \theta = \frac{\pi}{6}$$

Answer: $x = r\cos\theta = 2\sin\theta\cos\theta$ and $y = r\sin\theta = 2\sin^2\theta$. Then

$$\frac{dx}{d\theta} = 2(-\sin\theta\sin\theta + \cos\theta\cos\theta)$$

and

$$\frac{dy}{d\theta} = 4\sin\theta\cos\theta$$

For $\theta = (\pi/6)$, $\cos\theta = (\sqrt{3}/2)$ and $\sin\theta = (1/2)$ so $dx/d\theta = 1$ and $dy/d\theta = \sqrt{3}$ and the tangent slope is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sqrt{3}}{1}$$

10. Find the area of the region that is bounded by the given curve and lies in the specified sector

$$r = \sin\theta, \; \pi/3 \le \theta \le 2\pi/3$$

Answer: The area is given by the integral formula

$$\int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

which here is

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{2} (\sin\theta)^2 d\theta$$

using

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

the integral becomes

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{4}(1-\cos(2\theta))d\theta = \frac{1}{4}\theta - \frac{1}{8}\sin(2\theta)|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

which gives

$$\frac{1}{4}(\frac{2\pi}{3} - \frac{1}{2}\sin(\frac{4\pi}{3}) - \frac{\pi}{3} + \frac{1}{2}\sin(\frac{2\pi}{3}))$$