Math 6813 homework

Problems in parentheses are optional— not necessary, but you might find it interesting to think about them. You are responsible for the rest of the listed problems, that is, you should be able to do them if asked (say, on a midterm or final exam). Some are exercises that are not worth writing out in detail if you understand how to do them. Only the starred problems need to be turned in.

1. (complete by 9/2) Review the appendix on CW-complexes in the Hatcher text.

2. (due 9/11) 0.1 (Hint: Think of the torus minus a point as the square minus an interior point, with the sides identified in the usual construction of the torus. The identified sides form the two circles. I don't think it is necessary to have explicit formulas for the deformation, just understand what is going on.), 0.2, 0.3 (for 0.3 and many of the other problems, one uses the fact that if $g \simeq g'$ then $fg \simeq fg'$ and $gh \simeq g'h$ for any f and h for which these make sense), 0.4*, (0.5), (0.6a), 0.9* (Hint: there is an extremely short proof of this), 0.10*, 0.11*, (0.12)

3. (9/11) 0.14*, 0.17*

4. (9/23) 0.18, 0.19, 0.20*, 0.21*, 0.23

5. (9/23) (0.16 This one is optional.) Hint: Start with the CW-structure that has two n-cells e^n_+ and e^n_- in each dimension (the upper and lower hemispheres). Use the HEP to show that if $F\colon S^\infty \to S^\infty$ is a map with $F(e^0_+) = e^0_+$, then there is a homotopy R with $R_0 = F$, $R_t(e^0_+) = e^0_+$, and $R_1(e^n_+) = e^0_+$. Telescope some of these together to produce a homotopy from 1_{S^∞} to $c_{e^0_+}$.)

6. (9/23) 0.26* (Hint: The first step is to show that the inclusion $i\colon X\times\{0\}\cup A\times I \to X\times I$ is a homotopy equivalence. The argument uses the obvious deformation retraction from $X \times \{0\} \cup A \times I$ to $X \times \{0\}$, and the inclusion $j\colon X \times \{0\} \to X \times I$. After applying Corollary 0.20, the proof of Proposition 0.18 can be adapted. You don't need to fill in all its details for Proposition 0.18, but fill in some if you want.)

7. (10/7) 1.1.1, 1.1.2, 1.1.3* (first check that $\beta_g\beta_h^{-1}$ is conjugation by $[g*\overline{h}]$), 1.1.5*, 1.1.6*

8. (10/7) 1.1.8, 1.1.10, 1.1.11, 1.1.12*, 1.1.13* (the last words in the problem should be "in A is path homotopic to a path in A.")

9. (10/23) 1.1.15, 1.1.17, 1.1.18*, 1.2.1*

10. (10/23) 1.2.2*, 1.2.3* (induct on the number, for A_1 take the interior of an n-ball centered at p and disjoint from the other points)

11. (10/23) 1.2.4 (one way is to show that X deformation retracts to a wedge of circles)

12. (10/23) Study example 1.25 of Section 1.2 (the Hawaiian earring).

13. (10/23) 1.2.7*, 1.2.8*, 1.2.20* (Hint: To see that X is not homeomorphic to $\vee_\alpha S^1$, realize that $\vee_\alpha S^1$ is obtained from a disjoint union of circles by identifying the copies of $(1,0)$ in all the circles and taking the quotient topology. In the quotient topology, any subset that meeets each circle in a single point different from $(1,0)$ is closed.), 1.2.21 (Hint: $X * Y - X$ deformation retracts to Y, $X * Y - Y$ deformation retracts to X, and $X * Y - (X \cup Y) \approx X \times Y \times (0,1)$.)

14. (10/23) 1.2.6* (just adapt the proof of Prop 1.26), (1.2.14 is a nice problem, but a bit of experience with presentations is needed. The 3-generator, 3-relator presentation you get from attaching the 2-cells to the 1-skeleton can be manipulated into the form $\langle X, Y \mid XYXY = X^2 = Y^2 \rangle$, which is a well-known presentation for the quaternion group.)

15. (11/11) 1.3.1, 1.3.2, 1.3.4*, 1.3.5* (Hint: Lift (0,0), then lift the bottom edge, then lift the vertical edges using the Homotopy Lifting Property. The lifts of their top points converge to the lift of (0,1), so eventually lie in the preimage of an evenly covered neighborhood of (0,1).)

16. (11/11) 1.3.8* (you may assume the fact that every map $\tau\colon \widetilde{X} \to \widetilde{X}$ that is a lift of $p\colon \widetilde{X} \to X$ is a homeomorphism), 1.3.9*

17. (11/11)* Let $p\colon \widetilde{X} \to X$ be a covering space with X and $\widetilde{X}$ path-connected. Show that if $x, y \in X$, then $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality. Thus the statement that p is a k-sheeted covering does not depend on the choice of basepoint. Hint: Choose a path γ from x to y and use the lifts of γ to define a bijection $\varphi_\gamma\colon p^{-1}(x) \to p^{-1}(y)$.

18. (11/11) (1.3.6)

19. (11/11)* In the proof of the existence theorem, we defined the universal cover of (X, x_0) to be $\widetilde{X} = \{[\gamma] \mid \gamma\colon I \to X \text{ and } \gamma(0) = x_0\}$. Let $[\gamma] \in \widetilde{X}$ and define $\delta\colon I \to \widetilde{X}$ by $\delta(t) = [\gamma_t]$, where for $t > 0$, γ_t is $\gamma\big|_{[0,t]}$ reparameterized to be a path $\gamma_t\colon I \to X$, and $\gamma_0 = c_{x_0}$. Prove that δ is continuous.

20. (11/20)* Let X be the one-point union of two circles, with fundamental group $F(a,b)$. Find a 4-fold covering $p\colon \widetilde{X} \to X$ such that $H = \pi_1(\widetilde{X})$ has index 2 in $N(H)$.

21. (11/20)* Let X be the one-point union of two circles, with fundamental group $F(a,b)$. Find an 8-fold regular covering $p\colon \widetilde{X} \to X$ such that $H = \pi_1(\widetilde{X})$ contains a^2, b^2, and $(ab)^4$. Prove that H is the normal closure K of $\{a^2, b^2, (ab)^4\}$. (Hint: First observe that K is contained in H, so it suffices to show that K has index at most 8. For this, you may use the fact that the dihedral group D_4 of order 8 has a presentation $\langle x, y \mid x^2, y^2, (xy)^4\rangle$. This implies that there is a surjective homomorphism from D_4 to $F(a,b)/K$: There is a surjective homomorphism from $F(x,y)$ to $F(a,b)/K$ defined by sending x to a, y to b; since this sends x^2, y^2, and $(xy)^4$ to $K = 1$, it induces a surjective homomorphism from $D_4 = F(x,y)/ \ll x^2, y^2, (xy)^2 \gg$ to $F(a,b)/K$, where $\ll x^2, y^2, (xy)^2 \gg$ indicates the normal closure of these elements.)

22. (11/20) 2.1.1, 2.1.2, 2.1.3

23. (12/4) 2.1.4*, 2.1.5*, 2.1.6* (For this one, I assume that he means that on Δ_0^2 one identifies $[v_0, v_1]$ to $[v_1, v_2]$ and to $[v_0, v_2]$. I get a complex with $n+1$ 2-cells, $n+1$ 1-cells, one 0-cell, and homology groups $H_2(X) = 0$, $H_1(X) = \mathbb{Z}/2^n\mathbb{Z}$, and $H_0(X) = \mathbb{Z}$. Let me know if you are getting the same result. Hint for the calculations: if you have a homomorphism from $\mathbb{Z}^N$ to $\mathbb{Z}^N$ given by an integer matrix, you can change basis using row and column operations that add integer multiples of rows or columns, and interchanges of rows or of columns. Doing such operations allows one to simplify the matrix until it becomes easy to see its kernel and image.)

24. (12/4) 2.1.9* (Make the identification of each $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ and $[v_0, \ldots, \widehat{v}, \ldots, v_n]$ using the order-preserving correspondence of the vertices. The homology depends on whether n is odd or even.)

25. (12/4) 2.1.11, 2.1.12, 2.1.13* (I think one just needs to define $P\colon \mathbb{Z} \to C_0Y$ to be the zero homomorphism, then on C_0X one has $\partial P + P\epsilon = \partial P + P\partial = g_\# - f_\#$ as before.)

26. 2.1.15, 2.1.16, 2.1.17 (for part (b), you may assume the facts that $H_2(X) \cong \mathbb{Z}$, $H_1(X) \cong \mathbb{Z}^4$, $H_1(A) \to H_1(X)$ is zero, and $H_1(B) \to H_1(X)$ is is injective)

27. Consider a commutative diagram of abelian groups and homomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow \\
0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \longrightarrow & 0
\end{array}$$

with exact rows. Prove that if α and γ are isomorphisms, then β is also an isomorphism. Give a counterexample to the converse.

28. (this is based on problem 2.1.21) Let $\Sigma X = X \times [-1,1]/((x,t) = (y,u)$ if $t = u = -1$ or $t = u = 1)$ be the suspension of X. Put $c_{-1} = [X \times \{-1\}]$ and $c_1 = [X \times \{1\}]$. Observe that ΣX is the union of upper and lower cones C_1 and C_{-1} meeting at the points of $X = X \times \{0\}$. Justify each of the following isomorphisms: $\widetilde{H}_n(\Sigma X) \cong \widetilde{H}_n(\Sigma X, c_{-1}) \cong \widetilde{H}_n(\Sigma X, C_{-1}) \cong \widetilde{H}_n(\Sigma X - \{c_{-1}\}, C_{-1} - \{c_{-1}\}) \cong \widetilde{H}_n(C_1, X) \cong \widetilde{H}_{n-1}(X)$.