Research Statement

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My research is at the intersection of commutative algebra and algebraic geometry, including applications to statistics. In particular, I study singularities in positive characteristic, with applications to the study of symbolic powers (Section 2) and connections to representation theory (Question 3.2). On the statistics side, I have studied the maximum-likelihood degrees of toric varieties (Section 4).

1. Commutative algebra—the context

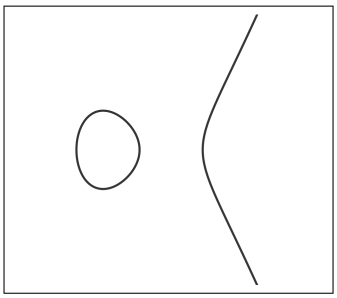

Figure 1. An algebraic variety

Algebraic geometers study geometric objects called *algebraic varieties* by associating them to algebraic objects called rings. For our purposes, algebraic varieties are the shapes obtained by plotting solutions to polynomial equations, as well as the shapes obtained by intersecting a collection of such plots. The corresponding ring is thought of as the set of regular functions defined on the variety in question.

A fundamental problem in algebraic geometry is that of classification: what are all the algebraic varieties out there? It was determined through work on the so-called Minimal Model Program that solving this classification problem requires careful study of singularities, meaning the cusps and self-intersections that appear in various algebraic varieties.

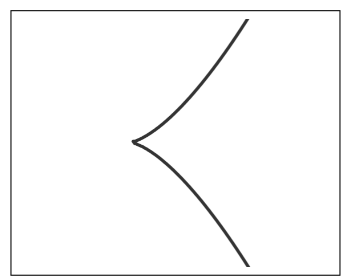

Figure 2. A singular algebraic variety

Most of this work has been done in the characteristic-zero setting, meaning that the solutions to the polynomial equations in question are taken over the complex numbers. Meanwhile, commutative algebraists have studied rings of positive characteristic. These rings come with the *Frobenius map*, which has numerous profound consequences.

While the positive-characteristic world is interesting in its own right, it also provides a powerful tool for answering questions about the characteristic-zero world. This is done using *reduction to positive characteristic*. To understand the method of reduction to positive characteristic, consider the integer 15 and its reductions modulo various primes

p	2	3	5	7	11	13	17	19	23	$\cdots$
15 mod p	1	0	0	1	4	2	15	15	15	$\cdots$

We see that we recover the original number, 15, when we consider its residues modulo sufficiently large prime numbers. There is a similar process for associating a positive-characteristic object to any algebraic variety. The philosophy of reduction to positive characteristic is that, to answer any question about a variety in characteristic zero, it should suffice to answer that question for reductions of that variety modulo any sufficiently large prime number. Once we mod out by a prime number, our question should become easier to answer, as we have the Frobenius map at our disposal.

My work in commutative algebra focuses on techniques in positive characteristic that can be used to study singularities of algebraic varieties. My motivation has been the problem of comparing ordinary and symbolic powers of ideals. An *ideal* is an algebraic object attached to a subvariety in a larger ambient variety—namely, the set of regular functions on the ambient variety that vanish along the subvariety. There are two notions of what it means to raise an ideal to an integer power. One notion, the *ordinary power*, comes from algebra. This operation is easily defined in terms of a generating set for the ideal, but is difficult to describe geometrically. On the other hand, the *symbolic power* of an ideal has a natural geometric interpretation while being difficult to compute algebraically. A big problem in commutative algebra is determining the relationship between these two notions. Using positive characteristic techniques, I have shown that a surprisingly large class of algebraic varieties behaves similarly to smooth varieties in this respect.

2. Test Ideals and Applications to Symbolic Powers

My research has focused on test ideals. Test ideals are an important tool for the study of singularities in positive characteristic. They have also had some surprising applications, such as the problem of comparing symbolic and ordinary powers of ideals [ELS01, HH02, Har05, MS17]. This is a compelling problem: an ideal I is the set of functions vanishing along some subspace of your variety, and its n-th symbolic power $I^{(n)}$ is the set of functions vanishing to order at least n along that subspace. Multiplying n functions that vanish to order 1 yields a function vanishing to order n, so we see that $I^n \subseteq I^{(n)}$. However, it happens that there can be other, strange functions vanishing to order n along the subspace. The broad question is: what do these strange functions look like? More concretely, Huneke, Katz, and Validashti asked: does there exist some factor h depending only on the ambient ring R such that

$$I^{(hn)} \subseteq I^n \tag{$\star$}$$

for all I and n [HKV09]?

Ein, Lazarsfeld, and Smith [ELS01] constructed a number h such that equation ($\star$) holds in the setting of regular rings, which are rings without any singularities. In particular, they show one can take h to be the *dimension* of the ambient ring in this setting. It is conjectured that this equation holds in broad generality, though few cases have been proven. Much of my work has been in this direction.

Ein–Lazarsfeld–Smith's proof depends on a property of test ideals known as subadditivity. This property says:

$$\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

when R is regular. Here, $\tau(R, \mathfrak{a}^s)$ denotes the test ideal of the data $(\mathfrak{a}, s)$, where $\mathfrak{a} \subseteq R$ is an ideal and $s > 0$. This subadditivity property turns out to be the only reason why regularity is required for the Ein–Lazarsfeld–Smith theorem. Thus, I was inspired to find a generalization of this subadditivity formula to the non-regular case that could be used to generalize the Ein–Lazarsfeld–Smith result. In [Smo18], I found such a formula:

Theorem 2.1 ([Smo18])**.** *Let k be a field of finite characteristic and let R be a k-algebra essentially of finite type. Then:*

$$\tau(R, \mathscr{D}^{(2)}, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

for any ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$ and any $s, t \geq 0$.

The correction term, $\mathscr{D}^{(2)}$, in the above formula is a subtle invariant of the ring R. Loosely speaking, it is a set of maps (i.e. a *Cartier algebra*) on R whose size witnesses subtle properties of the diagonal embedding of the corresponding algebraic variety, which in turn measures the singularities of R. In particular, if R is regular, then $\mathscr{D}^{(2)}$ is as large as possible. These connections remain to be fully explored.

This is not the first subadditivity formula for test ideals of singular rings. Earlier ones were given by Takagi in [Tak06] and Eisenstein in [Eis10]. We show in [Smo18] that this subadditivity formula is sharper than each of the previous formulas:

Theorem 2.2 ([Smo18])**.** *Let k be a field of finite characteristic and let R be a k-algebra essentially of finite type. Then:*

$$\operatorname{Jac}(R)\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathscr{D}^{(2)}, \mathfrak{a}^s\mathfrak{b}^t)$$

for any ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$ and any $s, t \geq 0$.

Theorem 2.3 ([Smo18])**.** *Let k be a field of characteristic 0 and let R be an equidimensional k-algebra essentially of finite type. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R. For p prime, let R_p denote the reduction of R modulo p, and likewise for $\mathfrak{a}_p, \mathfrak{b}_p$. Then*

$$\overline{\operatorname{Jac}(R_p)}\tau(R_p, \mathfrak{a}_p^s\mathfrak{b}_p^t) \subseteq \tau(R_p, \mathscr{D}^{(2)}, \mathfrak{a}_p^s\mathfrak{b}_p^t)$$

for all $p \gg 0$.

Having shown that this subadditivity formula is sharper than earlier ones, it makes sense to ask whether it yields new results on symbolic powers. It turns out that, if the set $\mathscr{D}^{(2)}$ and its generalizations $\mathscr{D}^{(n)}$ are sufficiently large (i.e. F-regular) for a ring R, then one can run Ein–Lazarsfeld–Smith's argument on that ring and recover the same result. In [CS18], we call such rings *diagonally F-regular* (we say a ring is n-diagonally F-regular if $\mathscr{D}^{(n)}$ is sufficiently large for that particular number n). Thus, diagonally F-regular rings R satisfy equation ($\star$) with $h = \dim R$. Further, we found that the Segre product of any two polynomial rings is diagonally F-regular, as well as the following:

Theorem 2.4 ([CS18])**.** *Let k be a field of positive characteristic and R, S two diagonally F-regular k-algebras. Then $R \otimes_k S$ is diagonally F-regular.*

This yields an infinite family of new examples of rings in which ($\star$) is known to hold. It also provides the first explicit bound on h for some examples where ($\star$) was only known to hold for some sufficiently large h.

Later, I extended these results with Janet Page and Kevin Tucker. We studied which *Hibi rings* are diagonally F-regular. A Hibi ring is a special kind of toric ring that can be associated to a finite partially ordered set. In [PST18], we find an infinite family of Hibi rings, described in terms of the combinatorics of their associated posets, which are diagonally F-regular:

Theorem 2.5 ([PST18])**.** *A Hibi ring is diagonally F-regular if the set of top nodes of the associated poset is totally ordered.*

This provides yet another new class of rings for which ($\star$) holds with $h = \dim R$. It is unknown whether or not the converse of this theorem holds, though we provide examples suggesting it might.

Another natural question is to ask how to compute the sets $\mathscr{D}^{(n)}$ for a given ring. This question appears to be quite difficult in general. However, one case where this is tractable is the case when R is a toric ring. In this case, I reduced the problem of computing $\mathscr{D}^{(n)}$ to a problem in polyhedral geometry. For instance, if $n = 2$, we have:

Proposition 2.6 ([Smo18], c.f. [PST18])**.** *Let $X = X(\Sigma)$ be a d-dimensional affine toric variety with anti-canonical polytope P_{-K}. Then $\mathscr{D}^{(2)}$ is generated in degree e, as a k-vectorspace, by the maps π_a where: $a \in \frac{1}{p^e}\mathbb{Z}^d$ and the interior of $P_{-K} \cap (a - P_{-K})$ contains a representative of each equivalence class in $\frac{1}{p^e}\mathbb{Z}^d/\mathbb{Z}^d$.*

Here, π_a is the map sending x^{u/p^e} to $x^{(a+u)/p^e}$ when $(a+u)/p^e$ is an integer and to zero otherwise.

Using these results, one can, in principle, determine whether or not a toric ring is diagonally F-regular by elementary means. Thus, the problem of determining which toric varieties are diagonally F-regular lends itself to good undergraduate research problems. I am currently mentoring an undergraduate at Utah, Dylan Johnson, who is working on just that.

3. Future directions

The above work shows that Frobenius methods can be a powerful tool for studying the relationship between symbolic and ordinary powers. The diagonal Cartier algebras $\mathscr{D}^{(n)}$, in particular, merit further exploration. Given how new they are, many basic questions remain open.

For instance, the theory of diagonal Cartier algebras can, as it stands, only be used to show that ($\star$) holds with $h = \dim R$. However, we expect that for most rings we will have to use $h > \dim R$. Further, the use of test ideals in our arguments above feels very ad hoc. So my first question is:

Question 3.1. Can we find a weakening of diagonal F-regularity that gives us ($\star$) with $h > \dim R$?

My next question has connections with the representation theory. Representation theorists in the '80s showed that, on complete *diagonally F-split* schemes, section rings of ample line bundles are generated in degree 1 [BK07]. As a consequence, every ample line bundle on such schemes is very ample. Further, the associated embedding into projective space is projectively normal. On the other hand, we show in [CS18] that 2-diagonal F-regularity is a strengthening of being diagonally F-split. In [PST18], we show that diagonal F-regularity is a strengthening of 2-diagonal F-regularity. Indeed, we show that even 3-diagonal F-regularity is a strengthening of 2-diagonal F-regularity, at least for toric varieties. So, we ask:

Question 3.2. What can be said about divisors on 2-diagonally F-regular rings? On diagonally F-regular rings?

Another project, which I am currently working on with Javier Carvajal-Rojas, is finding transformation rules for these diagonal Cartier algebras. For instance, we ask:

Question 3.3. What is the behavior of $\mathscr{D}^{(n)}$ under finite morphisms? In which contexts is $\mathscr{D}^{(n)}$ well-behaved under base-change?

We hope that this work will also provide a partial answer to Question 3.1, similarly to the techniques in [HKV15].

Theorem 2.2, along with the following, show that these diagonal Cartier algebras offer a new perspective on singularity theory orthogonal to existing homological and Frobenius techniques:

Theorem 3.4 ([Smo18]). *If R is regular, then $\mathscr{D}^{(n)}$ is as large as possible for all $n > 1$. Conversely, if $\mathscr{D}^{(n)}$ is as large as possible for some $n > 1$, then R is F-regular. If R is toric, $\mathbb{Q}$-Gorenstein, and $\mathscr{D}^{(n)}$ is as large as possible for some $n > 1$, then R is regular.*

Thus, one of my long-term goals is to study the following question:

Question 3.5. What information can one recover about the singularities of a ring R from its diagonal Cartier algebras $\mathscr{D}^{(n)}$?

I plan to answer this question by exploring the connection between diagonal Cartier algebras and the theory of differential operators, as both make heavy use of the diagonal embedding. Closely related to the theory of differential operators is the theory of differential forms. Part of the latter is various notions of the *different* of a ring extension, e.g. Noether differents, Kähler differents, and Dedekind differents. These differents are defined in a somewhat similar way to diagonal Cartier algebras, providing further evidence that there are treasures to be uncovered in studying this question.

Note that these diagonal Cartier algebras, and therefore the notion of diagonal F-regularity, are only defined for rings in positive characteristic. As with all other measures of singularity in positive characteristic, one hopes there is an analog in characteristic 0 defined in terms of resolutions of singularities:

Question 3.6. Characterize the diagonal F-regularity of mod-p reductions of a complex variety X in terms of its discrepancies.

Finally, I hope to break down the following question into smaller chunks that I can assign as undergraduate research projects:

Question 3.7. Which toric varieties are diagonally F-regular?

There are many related questions ripe for undergraduate research as well. For instance, we can further explore how $\mathscr{D}^{(n)}$ relates to $\mathscr{D}^{(m)}$, or whether n-diagonal F-regularity is independent of characteristic for toric rings.

4. Algebraic Statistics

My other research is in algebraic statistics. I was initiated into this new field of study when I participated in the Math Research Community session on algebraic statistics in the summer of 2016. Algebraic statistics is the study of applying algebraic geometry to solve problems in statistics. Our collaboration, consisting of nine people, studied a measure of complexity known as the *maximum likelihood (ML) degree* of discrete random models.

A fundamental problem in parametric statistics is finding the parameters of a given model that best fit a given data set. For instance, one might assume that the height of students in a classroom is normally distributed. One might then try to deduce the standard deviation of this hypothetical normal distribution given a sample of students' heights. One way of doing this is *maximal likelihood estimation*, where one uses the value of a parameter for which the probability (technically, the "likelihood") of obtaining the data one has is maximized. This is done by solving the so-called likelihood equations, which give local maxima for this likelihood. However, it is possible that there is more than one solution to the likelihood equations. It turns out that the number of these solutions is constant for generic data for a given model; this is called the maximum-likelihood degree of the model.

We studied the maximum-likelihood degree of discrete random models. A *discrete random model* is a discrete random variable $X(\theta_1, \ldots, \theta_d)$ whose probabilities p_i are polynomial functions of some parameters $\theta_j \in \mathbb{C}$. Thus, in the language of algebraic geometry, a discrete random model is the same as a regular map $(\mathbb{C}^*)^d \to \mathbb{P}^n$ where each coordinate is given by a monomial. The closure of the image of such a map is then a toric variety. Thus, solving the likelihood equations amounts to intersecting subvarieties of a toric variety. The ML-degree of any variety is known to be less than or equal to its degree. Using intersection theory, we found a general criterion for determining when this inequality is strict:

Theorem 4.1 ([ABB$^+$18])**.** *Let $V^c \subset \mathbb{P}^{n-1}$ be the projective variety defined by the monomial parametrization $\psi^c : (\mathbb{C}^*)^d \longrightarrow (\mathbb{C}^*)^n$ where*

$$\psi^c(s, \theta_1, \theta_2, \ldots, \theta_{d-1}) = (c_1 s\theta^{a_1}, c_2 s\theta^{a_2}, \ldots, c_n s\theta^{a_n}),$$

and $c \in (\mathbb{C}^*)^n$ is fixed. Then $\operatorname{mldeg}(V^c) < \deg(V)$ if and only if c is in the principal A-determinant of the toric variety $V = V^{(1,\ldots,1)}$.

We also found ad hoc formulas for the ML-degrees of important classes of toric varieties, such as Segre varieties and Veronese varieties. Further questions include expanding these ad hoc formulas, for instance computing the ML-degree of arbitrary Veronese varieties. This work is also fairly accessible and may prove fertile ground for undergraduate research projects.

I would also like to find more applications of multiplier ideals and test ideals in statistics. For instance, Watanabe [Wat99] found applications of the tools from birational geometry, such as the log-canonical threshold, to statistical learning theory. It stands to reason that there should be useful applications of the theory of F-singularities, and in particular the F-pure threshold, to this area as well. I would like to explore this further. Finding such an application would be beneficial, as F-pure thresholds are often easier to compute than their characteristic 0 counterparts. Indeed, I have helped write software that does so [BHK$^+$18].

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