An intro to D-modules

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1 The Weyl algebra

Let $K[x_1, \cdots, x_n]$ (henceforth abbreviated $K[\mathbf{X}]$) be a polynomial ring. The *nth Weyl algebra* is a K-subalgebra of $\mathrm{End}_K(K[\mathbf{X}])$ generated by:

- $\hat{x}_i$, where $\hat{x}_i(g) = x_i g$
- $\frac{\partial}{\partial x_i}$

Typically we denote $\hat{x}_i$ simply by x_i and $\frac{\partial}{\partial x_i}$ by ∂_i. So a typical element of A_2 might look like

$$\sigma = \partial_1^2 \partial_2 x_1 x_2 - 4x_1^3 x_2^2$$

where

$$\sigma(x_1^2) = \frac{\partial^3}{\partial^2 x_1 \partial x_2} x_1 x_2 x_1^2 - 4x_1^3 x_2^2 x_1^2 = \frac{\partial^3}{\partial^2 x_1 \partial x_2} x_1^3 x_2 - 4x_1^5 x_2^2 = 6x_1 - 4x_1^5 x_2^2$$

We denote this Weyl algebra A_n. Note that the Weyl algebra is **NOT** commutative! It comes with a commutator $[\cdot, \cdot]$ defined by $[a, b] = ab - ba$. Here are some commutators to know:

- $[\partial_i, x_i] = 1$, because for any $f \in K[\mathbf{X}]$, the product rule says
$$[\partial_i, x_i]f = (\partial_i x_i - x_i \partial_i) f = \partial_i(x_i f) - x_i(\partial_i f) = f + x_i \partial_i f - x_i \partial_i f = 1 \cdot f$$
- More generally, $[\partial_i, f] = \frac{\partial f}{\partial x_i}$ (this is an element of the Weyl algebra!)
- Thus, $[\partial_i, x_j] = \delta_{ij}$
- $[x_i, x_j] = 0$ for all i, j

1.1 Canonical form

Writing $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is clunky, so we adopt the following notation: given $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, by x^α we mean $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Similarly for ∂^α. Such an α is called a *multi-index*. Also, by $|\alpha|$ we mean $\alpha_1 + \cdots \alpha_n$. If char $k = 0$, The Weyl algebra A_n has K-vector space basis given by $\{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$. For instance,

$$\partial_i^2 x_i^2 = \partial_i(x_i^2 \partial_i + 2x_i) = \partial_i x_i^2 \partial_i + 2\partial_i x_i = (x_i^2 \partial_i + 2x_i)\partial_i + 2x_i \partial_i + 2$$

What goes wrong if char $k = p$? Well, in that case $\partial_i^p = 0$! Indeed, if $a \geq p$, then $\partial_i^p x_i^a = 0$.

This canonical form is our friend and most proofs will rely on looking at it and applying some commutators cleverly.

1.2 Degree of an operator

Given a monomial $x^\alpha \partial^\beta \in A_n$, we can define $\deg x^\alpha \partial^\beta = |\alpha| + |\beta|$. Given any element of A_n, we define its degree to be the largest degree among the degrees of its monomials. It takes a bit of work, but one can show

Proposition 1.1. *We have the following:*

- $\deg(D + D') \leq \max\{\deg D, \deg D'\}$
- $\deg(DD') = \deg D + \deg D'$
- $\deg[D, D'] \leq \deg D + \deg D' - 2$

Proof. 1 is clear. For 2 and 3, Induce on $degD + degD'$. Also, from 1, it's enough to assume that D and D' are monomials. □

This shows that A_n is a domain. Also, we have the following neat fact:

Proposition 1.2. *A_n has no nontrivial two-sided ideals (so it's a simple algebra)*

Proof. Proof by contradiction. Suppose I is a nonzero two-sided ideal. Choose D of minimal degree k in I. Then $[D, f] \in I$ for all f. Now, $\deg D > 0$, so there is some α and β such that $x^\alpha \partial^\beta$ has a nonzero coefficient in the expansion of D, say $\beta_i > 0$. But then $[x_i, D]$ is nonzero and has degree $\leq k - 1$. Thus, $\beta = 0$. But if $\alpha_i > 0$, then $[D, \partial_i]$ has degree $\leq k - 1$ and is not zero. □

2 Modules over A_n

We haven't even defined a D-module yet! A D-module is a module over A_n (or the ring of differential operators of any ring).

3 Graded and filtered modules

Recall that a ring R is graded if $R = \bigoplus_i R_i$ with $R_i \cdot R_j \subseteq R_{i+j}$. If R is a graded ring, then a *graded R-module* is a module M such that $M = \bigoplus_i M_i$ where $R_i M_j \subseteq M_{i+j}$.

Now, we'd like to make A_n into a graded algebra. There's one problem though: we can't say something like "$\partial_1 x_1$ degree 2" because $\partial_1 x_1 = x_1 \partial_1 + 1$. So instead we have to be content with a filtration. Recall that a *filtration* of a K-algebra R is an ascending chain of K-vector spaces $F_0 \subset F_1 \subset F_2 \ldots$ such that $F_i F_j \subset F_{i+j}$.

One filtration on A_n that's of particular interest to us is the *Bernstein filtration*, denoted $\mathcal{B}_i$. We define $\mathcal{B}_i$ to be the k-vectorspace generated by $\{x^\alpha \partial^\beta \mid |\alpha| + |\beta| \leq i\}$. This filtration is cool because every $\mathcal{B}_i$ is a *finite-dimensional* vector space over k.

One more construction: let R be a ring and $\{F_i\}$ a filtration of R. For all n, let $\sigma_n : F_n \to F_n/F_{n-1}$ be the usual quotient map. This map σ is called the *symbol* map in this context. Then the *graded ring associated to F*, denoted $\mathrm{gr}^F(R)$, is given by

$$\mathrm{gr}^F(R) = \bigoplus_i F_i/F_{i-1}$$

Note that any homogenous element of degree k in this ring can be written as $\sigma_k(q)$ for some $q \in F_k$.

In the case of the Weyl algebra, we write S_n to denote $\mathrm{gr}^{\mathcal{B}} A_n$.

Lemma 3.1. *S_n is a polynomial ring in $2n$ variables.*

Proof. It's easy to see that $\mathrm{gr}^{\beta} A_n$ is generated by $\sigma_1(x_i)$ and $\sigma_1(\delta_i)$ over k. Let $y_i = \sigma(x_i) \in S_n$ and $y_{n+i} = \partial_i \in S_n$. We wish to show that the y_i commute. For this it's enough to show that $y_i y_{n+i} = y_{n+i} y_i$. But this is clear: $\sigma_2(\partial_i x_i) = \sigma_2(x_i \partial_i + 1) = \sigma_2(x_2 \partial_i)$.

Thus we can write a surjective map from a polynomial ring $k[z_1, \cdots, z_{2n}] \to S_n$ by $z_j \mapsto y_j$, for $1 \leq j \leq 2n$. It remains to show that this map, φ, is injective. To that end, suppose some homogenous polynomial F is sent to zero. If $\varphi(F) = 0$, then $\varphi(F) = \sigma_k(d)$ where $d = F(x_1, \cdots, x_n, \partial_1, \cdots, \partial_n)$. But if $\sigma_k(d) = 0$, then d can also be written as a sum of monomomials of degree less than k. So d must be zero to begin with, since canonical form is a k-basis of A_n. □

Similarly define a filtered module and graded module associated to a filtration: if M is an A_n module, then a filtration of M with respect to $\mathcal{B}$ is an ascending chain of k-vector spaces $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ with $M = \bigcup_i \Gamma_i$ and $\mathcal{B}_i \Gamma_i \subseteq \Gamma_{i+j}$.

A filtration on M induces filtrations on its submodules and quotients: let Γ be a filtration on M with respect to $\mathcal{B}$ and let $N \subseteq M$ be a submodule. Then $\Gamma_i \cap N$ and $\Gamma_i + N/N$ are filtrations on N and M/N, repesctively, that agree with $\mathcal{B}$.

Now, let M be a left A_n module. Then

Lemma 3.2. *if $gr^{\Gamma} M$ is a (left) noetherian S_n module, then M is a noetherian A_n module*

Proof. Pick any $N \subseteq M$ and let Γ' be the induced grading on N. Then $\mathrm{gr}^{\Gamma'} N \subseteq \mathrm{gr}^{\Gamma} M$ is a finitely generated submodule. We wish to show that N is a finitely generated submodule of M.

Since $\mathrm{gr}^{\Gamma'} N$ is finitely generated, we can enumerate its generators $f_1, \cdots, f_s$. Then there is some m such that $\mathrm{gr}^{\Gamma'} N$ is generated by elements of degree $\leq m$. I claim that N is generated by elements of degree $\leq m$. Suppose this weren't the case: then choose a homogenous element $f \in N$ of minimal degree such that $f \notin A_n \cdot \Gamma'_m$. Then f has some degree $k > m$. Then $\sigma_k(f) = \sum \sigma_{k-r_i}(a_i) f_i$ for some $a_i \in S_n$. But this implies $f - \sum a_i \hat{f}_i \in \ker \sigma_k = \Gamma_{k-1}$, where $f_i = \mu(\hat{f}_i)$. This contradicts the fact that $\deg f = k$.

Thus N is generated by elements in Γ'_m. But Γ'_m is a finite-dimensional k-vector space with a finite dimensional basis. So this basis generates N.

□

The converse doesn't hold. If $\mathrm{gr}^{\Gamma} M$ is noetherian, we call Γ a *good* filtration of M. Theres a lemma saying all good filtrations are sorta the same. Namely:

Lemma 3.3. *A filtration Γ of M is good if and only if there exists some k_0 such that $\Gamma_{i+k} = \mathcal{B}_i \gamma_k$ for all $k > k_0$.*

and

Lemma 3.4. *Let Γ and Ω be two filtrations of M. If Γ is good, then there is some k_1 such that $\Gamma_j \subseteq \Omega_{j+k_1}$ for all j.*

Proof. This follows from the above: choose some k_0 such that $\Gamma_{i+k_0} = \mathcal{B}_i \Gamma_{k_0}$. Then there is some k_1 such that $\Omega_{k_1} \supseteq \Gamma_{k_0}$, since Γ_{k_0} is a finite-dimensional k-vector space. Then for any j,

$$\Gamma_j \subseteq \Gamma_{j+k_0} = \mathcal{B}_j \Gamma_{k_0} \subseteq \mathcal{B}_j \Omega_{k_0} \subseteq \Omega_{k_0+j}$$

□

Note that any Noetherian A_n module has a good filtration: if M is generated by $u_1, \cdots u_k$, set $\Gamma_j = \sum_{i=1}^k \mathcal{B}_j u_i$.

4 Dimension and multiplicity

Here's a result from commutative algebra:

Theorem 4.1. *Let $M = \bigoplus_{i\geq 0} M_i$ be a graded module over a polynomial ring $K[x_1, \cdots, x_n]$. There exists a polynomial $h(t) \in \mathbb{Q}[t]$ such that*

$$\sum_{i=0}^{s} \dim_k M_i = h(s)$$

for $s >> 0$.

Let M be a module over A_n and let h be the hilbert polynomial of $\mathrm{gr}^\Gamma M$, where Γ is a good filtration of M with respect to A_n. We define the *dimension* of M to be the degree d of h, and we define the multiplicity of M to be $d!a_d$, where a_d is the leading coefficient. It's not hard to see this is indepedent of good filtration using 4.3.

Example: $d(K[x_1, \cdots, x_n] = n$ and $d(A_n) = 2n$.

These have some nice properties:

Lemma 4.2. *Let M be a finitely generated A_n module and N a submodule. Then*

- $\dim M = \max\{\dim N, \dim M/N\}$
- *More generally, if $M = M_1 \oplus \cdots \oplus M_s$, then* $\dim M = \max\{\dim M_1, \cdots, \dim M_s\}$
- $m(M) = m(N) + m(M/N)$ *if* $\dim N = \dim M/N$

Proof. Let Γ be the good filtration of M. Let Γ' and Γ'' be the induced filtrations on N and M/N, respectively. It's easy to see that

$$0 \to \mathrm{gr}^{\Gamma'} N \to \mathrm{gr}^{\Gamma} M \to \mathrm{gr}^{\Gamma''} M/N \to 0$$

whence Γ' and Γ'' are good filtrations. This allows us to see $h_N + h_{M/N} = h_M$. The lemma follows. □

Now let M be a finitely generated A_n module. Since we have a surjection $A^n \to M$, the above lemma tells us that $\dim M \leq \dim A_n = 2n$. Surprisingly, we get another inequality:

Theorem 4.3 (Bernstein's inequality)**.** *If M is a finitely-generated A_n-module, $n \leq \dim M \leq 2n$*

Proof. First, we define a map $B_i \to \mathrm{Hom}_k(\Gamma_i, \Gamma_{2i})$ where b get's sent to "multiplication by b". This is an injective map. The proof uses induction and some manipulation of canonical forms. Indeed, to see this, it's enough to show that $a\Gamma_i \neq 0$ for any $0 \neq a \in \mathcal{B}_i$. We proceed by induction. The base case is easy since $\mathcal{B}_0 = k$.

Now, if $a\Gamma_i = 0$, then $a \notin K$, and hence the canonical form of a has some term $cx^\alpha \partial^\beta$ where $c \neq 0$, and $|\alpha| + |\beta| > 0$. Then $[a, \partial_i]$ is not zero and it's in $\mathcal{B}_i$. Now,

$$[a, \partial_i]\Gamma_{i-1} = a\partial_i\Gamma_{i-1} - \partial_i a\Gamma_{i-1}$$

but $a\Gamma_{i-1} = 0$ because $a\Gamma_i = 0$ and Γ_i contains Γ_{i-1}. Also, $\partial_i \Gamma_{i-1} \subseteq \Gamma_i$ So,

$$[a, \partial_i]\Gamma_{i-1} = 0$$

but this contradicts the induction hypothesis. We get that $\dim B_i \le \dim \mathrm{Hom}(\Gamma_i, \Gamma_{2i}) = \chi(i)\chi(2i)$. So $\chi(i)\chi(2i)$ is a polynomial of degree as least $2n$ in i. But, the degree of this polynomial is also $2d(M)$, so $d(M) \ge n$. $\square$

5 Holonomic modules

... are D-modules of minimal dimension.

It's easy to see holonomic modules are artinian, using multiplicity! Indeed, the length of a holonomic D-module cannot exceed its length.

6 Differential operators (time permitting)

Let R be a commutative k-algebra. Then we define the *differential operators of order* $\le n$ as follows:

- the differential operators of order 0 are the elements of End R whose commutator with anythign in R is zero
- order $\le n$ if $[a, P]$ has order $\le n-1$ for all $a \in R$.

operators of order 1 turn out to just be the derivations.

The Weyl algebra is $D(k[x_1, \cdots, x_n])$

6.1 Other construction of the Weyl algebra

It turns out the commutator relations above characterize the Weyl algebra. To be more precise, let $R = K\{z_1, \cdots, z_{2n}\}$ be the free (non-commutative!) K-algebra in $2n$ generators. Then we get a surjective map $R \to A_n$ given by $z_i \mapsto x_i$ for $i \le n$, and $z_{n+j} \mapsto \partial_j$ for $j \le n$. If char $k = 0$, the kernel of this map is exactly the two-sided ideal generated by

- $[z_i, z_j]$, $|i-j| \ne n$
- $[z_{n+i}, z_i] = 1$

Otherwise, the kernel includes z_{n+i}^p for all i. So there are *two* possible ways to define the Weyl algebra in characteristic p! I think the preference is to use something called "divided powers"... or just the ring of differential operators!