FINDING DIAGONALLY F-REGULAR TORIC RINGS

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1. Background

An important question in commutative algebra is undering the relationship between *symbolic* and *ordinary* powers of ideals. Work by D. Smolkin and J. Carvajal-Rojas shows that, for so-called *Diagonally F-regular* rings, the relationship between symbolic and ordinary powers is as nice as one could hope. None of these definitions are super important to understand for now but we just wanted to give you some background of what we're actually computing and why.

1.1. Localization.

Through out this section (and this whole writeup) assume that R is a commutative, noetherian ring with identity. One of the most powerful/important/ubiquitous tools in commutative algebra is localization. One way to think about localization is to think of it as adding certain multiplicative inverses to a ring.

Definition 1.1. A *multiplicatively closed set* (often called a multiplicative set) is a subset of a ring which is multiplicatvely closed and contains 1.

Exercise 1.1. Prove that for any ring R and prime ideal $\mathfrak{p} \subset R$, the set $R \setminus \mathfrak{p}$ is a multiplicative set.

Given a ring R and a multiplicative set S, we form a new ring $S^{-1}(R)$ as follows. We start with the product (as sets) $R \times S$. In order to make this into a ring, we are going to mod out by an equivalence relation. We will say that two elements (a, s) and (b, t) are equivalent in $R \times S$ if and only if there exists some element u in S such that $(at - bs)u = 0 \in A$. ‘

Exercise 1.2. Check that above relation is actually an equivalence relation.

Let a/s denote the equivalence class of (a, s). Then $S^{-1}(R)$ is the set of equivalence classes in $S \times R$. We can then put a ring structure on $S^{-1}(R)$ using the natural operations

$$a/s + b/t = (at + bs)/st$$
$$(a/s)(b/t) = ab/st$$

Exercise 1.3. Check this is actually a ring.

Exercise 1.4. There is a natural map $\iota : R \to S^{-1}(R)$ sending each $x \in R$ to $x/1$. Check that this map is a homomorphism of rings. Check also that this map is injective when R is an integral domain. Thus we can think of R as a subring of $S^{-1}(R)$.

Let $I = (f_1, \dots, f_r)$ be an ideal in R. This induces an ideal $(f_1/1, \dots, f_r/1)$ in $S^{-1}(R)$. This ideal is usually denoted $IS^{-1}(R)$ and called the *exension of I under the map ι*. It is the smallest ideal in $S^{-1}(R)$ that contains $\iota(I)$.

Now let R be an integral domain, so that $R \subseteq S^{-1}(R)$. Let J be an ideal in $S^{-1}(R)$. Then it makes sense to consider the intersection $J \cap R$. This called the *contraction* of J to R.

Exercise 1.5. Check that $J \cap R$ is an ideal of R. For any ideal $I \subseteq R$, check that $I \subseteq (IS^{-1}(R)) \cap R$.

For prime ideals $\mathfrak{p}$ we define $R_\mathfrak{p}$ to be $(R \setminus \mathfrak{p})^{-1}(R)$. In some sense we have inverted all of the elements away from $\mathfrak{p}$.

Example 1.2. Let k be your favorite field. Let $R = k[x]$ and $\mathfrak{p} = (x)$. Let's see if we can understand what $R_\mathfrak{p}$ looks like. The first step is to understand $(R \setminus \mathfrak{p})$. Since we can characterize $\mathfrak{p}$ as the set $\{f \in R | f(0) = 0\}$ we can characterize $(R \setminus \mathfrak{p}) = \{f \in R | f(0) \neq 0$. Thus $R_\mathfrak{p} = \{f/g | f, g \in R \text{ and } g(0) \neq 0\}$. This is the same as the set of rational functions in x that are defined at 0.

Exercise 1.6. Let $R = k[x, y]$. Let $\mathfrak{m} = (x, y)$. Find $R_\mathfrak{m}$.

1.2. **Ordinary and Symbolic Powers of an Ideal.** We start by defining two different operations on ideals.

Definition 1.3. Let I be an ideal of R. We define the nth *ordinary power* of an ideal, denoted I^n to be the ideal generated by all products of n elements of I. Note that, when $m > n$, we have $I^m \subseteq I^n$.

Example 1.4. Let $R = k[x, y], I = (x, y)$. Then $I^2 = (x^2, xy, y^2)$.

Definition 1.5. Let R be an integral domain and $\mathfrak{p} \subseteq R$ a prime ideal. We define the nth *symoblic power* of $\mathfrak{p}$, denoted $\mathfrak{p}^{(n)}$ to be the ideal

$$\mathfrak{p}^{(n)} := (\mathfrak{p}^n R_\mathfrak{p}) \cap R$$

Exercise 1.7. Show that, when $m > n$, we have $\mathfrak{p}^{(m)} \subseteq \mathfrak{p}^{(n)}$.

Remark 1.6. One can make sense of the symbolic powers $I^{(n)}$ of ideals $I \subseteq R$ even when I is not prime. But, this is a little more complicated and not much more interesting.

Example 1.7. Let $R = k[x, y]$ and $I = (x, y)$. Then $I^{(2)} = (x^2, xy, y^2)$. In fact, $I^{(n)} = I^n$ for all n in this case. We described R_I above, and it shouldn't be too difficult to convince yourself that the only things in $I^n R_I$ that are also in R are exactly the elements of I^n.

It follows from exercise 1.5 that $\mathfrak{p}^n$ is always contained in $\mathfrak{p}^{(n)}$. A big question in commutative algebra is how far we can expect this containment to be from an equality. In the above example, we have $I^n = I^{(n)}$, though this is certainly not always the case. The following property of rings is of interest for a variety of reasons—it says that, in some sense, the difference between $\mathfrak{p}^{(n)}$ and $\mathfrak{p}^n$ varies *uniformly* among all the different ideals $\mathfrak{p} \subseteq R$.

Definition 1.8. A ring R is said to have the Uniform Symbolic Topology Property if there exists h such that for all prime ideals $\mathfrak{p}$ of R and all $n > 0$,

$$\mathfrak{p}^{(hn)} \subset \mathfrak{p}^n$$

The goal of our project is to understand which toric varieties have this property!

1.3. **Toric Varieties.** Toric varieties are dope and hopefully you were able to get at least a little bit out of the book we sent you. Here's a brief crash course in the minimum you have to know to get started on the project.

Given a finite collection of vectors $v_i \in \mathbb{R}^n$ and any field k, one can associate a ring R by taking the subring of $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by monomials with the exponent vectors $\lambda \in \mathbb{Z}^n$ such that $\langle \lambda, v_i \rangle \geq 0$. In other words, the monomial $\underline{x}^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ is an elemlent of R if and only if $\langle \lambda, v_i \rangle \geq 0$ for all i. This ring R is called the *toric ring* (or, if you want to be fancy, the *affine toric variety*) associated to the vectors v_i.

Example 1.9. Take the vectors $(1, 0)$ and $(0, 1)$ then the associated toric ring is $k[x, y]$.

Example 1.10. Take the vectors $(-1, 2)$ and $(0, 1)$ then the toric ring is $k[y, xy, x^2y]$.

That's pretty much it. You're an expert on toric varieties now. I'm a great teacher.

2. Project

Given a collection of vectors $v_i \in \mathbb{R}^n$, one can associate an affine toric variety R. Then the *anticanonical polytope* of R is the set

$$P_R = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle > -1\}$$

An important question in commutative algebra is about the relationship between *symbolic* and *ordinary* powers of ideals. Work by D. Smolkin and J. Carvajal-Rojas shows that, for so-called *Diagonally F-regular* rings, the relationship between symbolic and ordinary powers is as nice as one could hope. Further, they show that for toric varieties, diagonal F-regularity can be checked in a purely combinatorial way, as follows: suppose char $R = p$ fix some $e > 0$. Then they define the set

$$\mathscr{D}_e^{(2)} = \left\{ v \in P_R \cap \frac{1}{p^e}\mathbb{Z}^n \mid \forall x \in \frac{1}{p^e}\mathbb{Z}^n,\ \exists y \in \mathbb{Z}^n : x + y \in P_R \cap (v - P_R)) \right\}$$

or, more generally, for any $m \geq 2$,

$$\mathscr{D}_e^{(m)} = \left\{ v \in P_R \cap \frac{1}{p^e} \mid \forall x_1, \ldots x_{m-1} \in \frac{1}{p^e}\mathbb{Z}^n, \exists y_1, \ldots y_m \in \mathbb{Z}^n : x_i + y_i \in P_R, \sum_i (x_i + y_i) \in v - P_R \right\}$$

Then the authors showed that R is diagonally F-regular whenever the sets $\bigcup_e \mathscr{D}_e^{(m)}$ are sufficiently large for all m (roughly speaking).

It's easier and probably no less interesting to consider consider the following set $\mathscr{D}^{(m)}$ instead of the sets $\mathscr{D}_e^{(m)}$ for various e:

$$\mathscr{D}^{(m)} = \left\{ v \in P_R \mid \forall x_1, \dots x_{m-1} \in \mathbb{R}^n, \exists y_1, \dots y_m \in \mathbb{Z}^n : x_i + y_i \in P_R, \sum_i (x_i + y_i) \in v - P_R \right\}$$

We want you to try to understand the sets $\mathscr{D}^{(m)}$ and hopefully find new examples of diagonally F-regular toric rings. Doing so would certainly be noteworthy within the commutative algebra community. In particular, we have broken down this project into the following steps:

(1) For a few specific toric rings, describe the set $\mathscr{D}^{(2)}$ as a union of explicit polytopes in $\mathbb{R}^n$. Perhaps a good place to start would be to understand the the toric ring associated to $(1, 0)$ and $(-a, b)$ for $a, b \in \mathbb{N}$.
(2) Figure out for which of the above toric rings we have that $\mathscr{D}^{(2)}$ is "sufficiently large" in the above sense.
(3) Find sufficient conditions for a toric ring that ensures $\mathscr{D}^{(2)}$ is sufficiently large.
(4) Repeat steps (1)-(3) for $\mathscr{D}^{(m)}$ instead of $\mathscr{D}^{(2)}$.