Studying Singularities via Reduction modulo p

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Setup: what is algebraic geometry?

Algebraic geometers study geometric objects called **algebraic varieties** by associating them to algebraic objects called rings. Historically, algebraic varieties were defined to be shapes obtained by plotting solutions to polynomial equations, as well as the shapes obtained by intersecting a collection of such plots. The modern definition of algebraic varieties is more general, but this historical definition is a good source of examples and intuition.

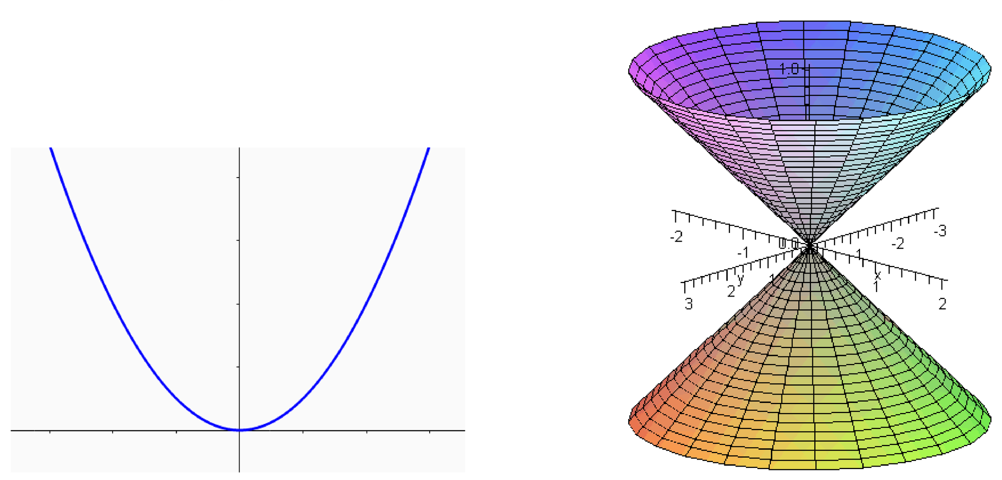

Figure 1: Two examples of algebraic varieties. Left: $y = x^2$. Right: $z^2 = xy$.

Birational classification

A fundamental problem in algebraic geometry is classification: what are all the algebraic varieites out there? As with any classification problem in mathematics, one must first establish a notion of equivalence: when do we consider two algebraic varieites to be the same? One notion, called birational equivalence, was devised by Riemann in his 1851 thesis. Two varieties are said to be **birationally equivalent** if, roughly speaking, you can bend one varity into the other after removing a few points. Thus, what we're interested in is technically the question of **birational classification**: what are all the algebraic varieties out there, up to birational equivalence?

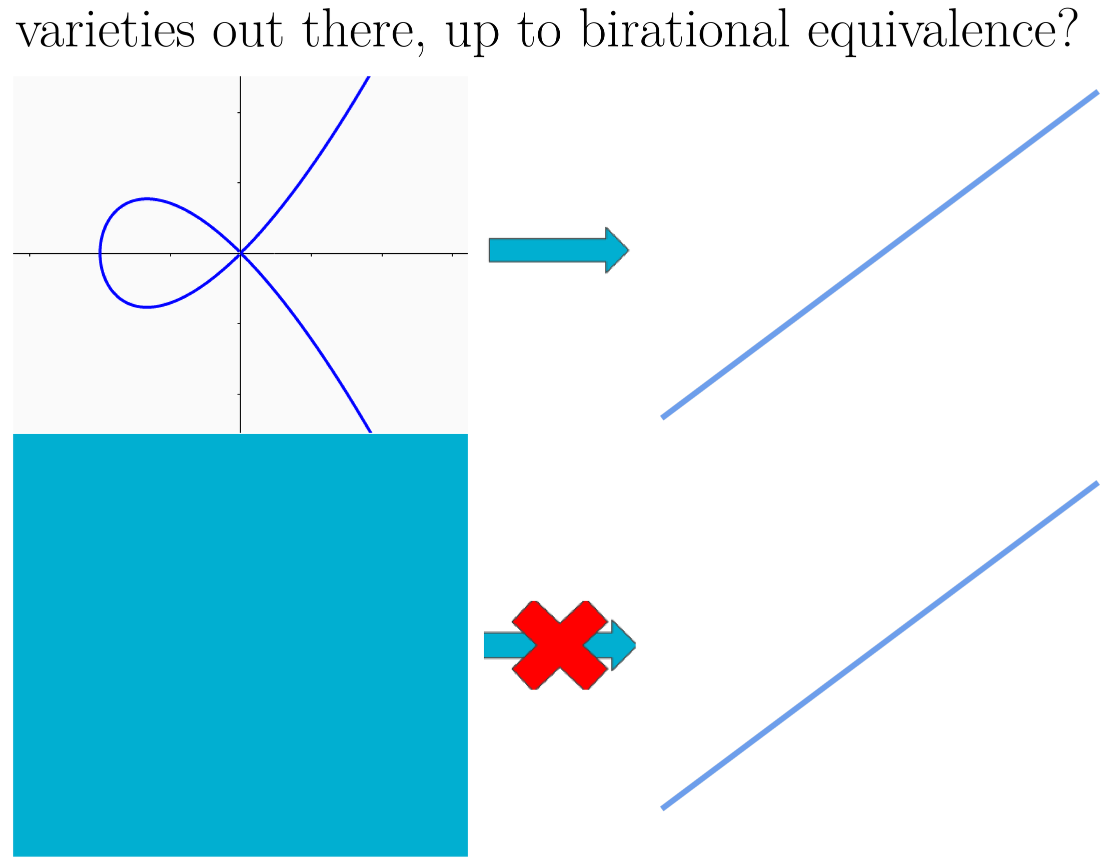

Figure 2: The top two varieties are birationally equivalent. The bottom two are not

Singularities

It was determined through work on the so-called Minimal Model Program that solving the birational classification problem requires careful study of singularities. A **singularity** in an algebraic variety is a spot where the variety intersects itself or has a pinch point. This led to the study of singularities of pairs and a heirarchy of singularities used today, such as log-terminal, rational, log-canonical, and so on. Most of this work has been done in the **characteristic-zero** setting, meaning, for instance, that the solutions to the polynomial equations in question are taken over the complex numbers.

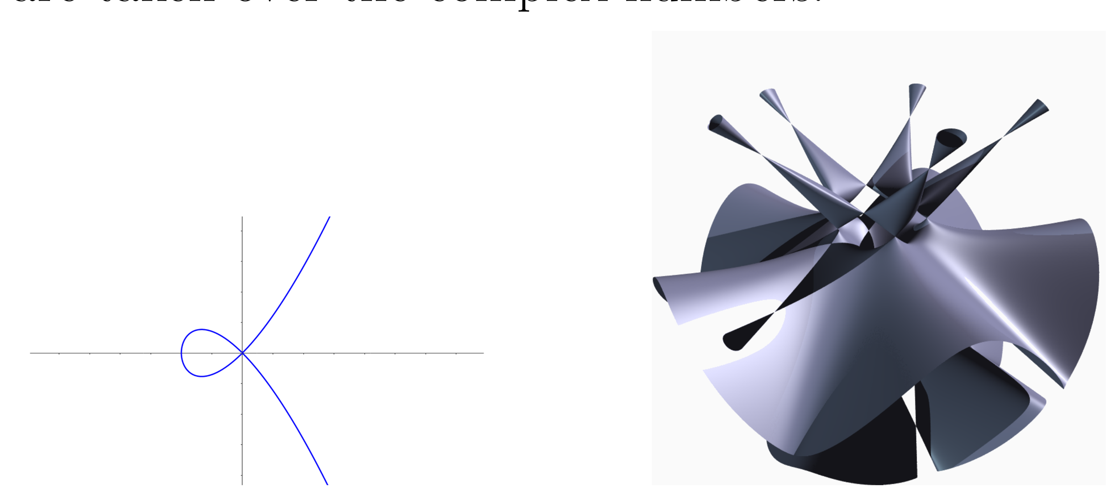

Figure 3: Each of these varities has some singularities

Reduction modulo p

Meanwhile, others commutative algebraists have studied rings of **positive characteristic**: rings where adding 1 to itself some amount of times (preferably a prime-number amount) yields 0. One example is the ring of integers modulo 3: $1+1+1 \equiv 0 \mod 3$. Rings of positive characteristic can be in many ways simpler than rings of characteristic zero. This means that, while the positive-characteristic world is interesting in its own right, it also gives a powerful tool for answering questions about the characteristic-zero world. This is done using **reduction to positive characteristic**. To understand the method of reduction to positive characteristic, consider the integer 15 and its reductions modulo various integers:

n	1	2	3	4	5	10	15	16	17	18
$15 \mod n$	0	1	0	3	0	5	0	15	15	15

We recover the original number, 15, by considering its residues modulo sufficiently large integers. There is a similar process for associating a positive-characteristic object to any algebraic variety. The philosophy of reduction to positive characteristic is that, to answer any question about a variety in characteristic zero, it should be sufficient to answer that same question modulo any sufficiently large prime number.

Multiplier ideals/Test ideals

This philosophy of reduction to positive characteristic has proven useful for studying singularities relevant to the birational classification problem. Indeed, Hochster and Huneke came up with a framework for studying singularities in positive characteristic in their seminal work on tight closure. Remarkably, this framework corresponds via reduction to positive characteristic to the tools that existed in characteristic 0. For example, my research has focused on **test ideals**: the test ideal of a ring R of positive characteristic provides a subtle measure of the singularities of its corresponding variety. The multiplier ideal, defined for rings of characteristic zero, plays a similar role. It has been shown that the multiplier ideal corresponds to the test ideal under that process of reduction to positive characteristic.

My research

My research has been focused on test ideals. Test ideals behave really well for rings without singularities. In that setting, they satisfy an important property known as **subadditivity**:

$$\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

My thesis is about finding a similar formula for rings that do have singularities. I came up with a candidate for such a formula, as did some other mathematicians before me. However, I proved the following theorem, which shows my formula is stronger than an earlier one:

Theorem

Let k be a field of characteristic 0 and let R be an equidimensional k-algebra essentially of finite type. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R. For p prime, let R_p denote the reduction of R modulo p, and likewise for $\mathfrak{a}_p, \mathfrak{b}_p$. Then

$$\overline{\mathrm{Jac}(R_p)}\tau(R_p, \mathfrak{a}_p^s\mathfrak{b}_p^t) \subseteq \tau(R_p, \mathcal{C}^{\mathrm{diag}}, \mathfrak{a}_p^s\mathfrak{b}_p^t)$$

for all $p \gg 0$.

References

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