Symbolic powers in rings of positive characteristic

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1 Learning objectives

1. We can understand the symbolic powers of (positive characteristic) rings by closely studying the maps $R^{1/p^e} \to R$.
2. For certain rings (e.g. Toric varieties, Hibi rings) the study of these maps boils down to (hard!) combinatorics

2 Symbolic and ordinary powers of ideals

We assume, for simplicity:

Global assumptions: R is a normal domain finitely generated over a perfect field k.

Though everything works even if k is not perfect and R is just reduced.

3. **Definition:** if $\mathfrak{p} \in \operatorname{Spec} R$, we define $\mathfrak{p}^{(n)} :=$ ________________________
4. Remark: these are larger than ordinary powers, i.e. $\mathfrak{p}^{(n)} \supseteq \mathfrak{p}^n$. Rarely an equality.
5. **Exercise:** let $R = k[x, y, z]/(xy - z^5)$ and $\mathfrak{p} = (x, z)$. Then $\mathfrak{p}^{(5)} =$ __________ $\supsetneq \mathfrak{p}^5$.
6. **Exercise:** Let $\mathfrak{m} \subseteq R$ be a maximal ideal. Then $\mathfrak{m}^{(n)} = \mathfrak{m}^n$ for all n.
7. Intuition: $\mathfrak{p}^{(n)}$ is the set of regular functions on $\operatorname{Spec} R$ that ____________________________________ __________________________ (*cf.* Zariski-Nagata theorem).

Main question: How does $\mathfrak{p}^{(n)}$ relate to $\mathfrak{p}^n$? More precisely, for which $a, b \in \mathbb{N}$ do we have ______________________ ?

8. In 2000, Ein, Lazarsfeld, and Smith gave a striking answer to this question:

   Theorem 1 ([ELS01]). *Let R be a regular ring over an algebraically closed field of characteristic 0. Then $\mathfrak{p}^{(hn)} \subseteq \mathfrak{p}^n$ for all prime ideals $\mathfrak{p}$ of height h.*

9. We will talk about weakening the regularity assumption in this theorem.
10. Remark: in particular, if $\dim R = d$, we see that $\mathfrak{p}^{(dn)} \subseteq \mathfrak{p}^n$ for all $\mathfrak{p}$ and all n. Because this number d depends only on the ring R (and not on the primes $\mathfrak{p}$) we say these rings have the *Uniform Symbolic Topology Property*, or USTP for short.

3 Commutative algebra mod p

11. To prove something like Theorem 1, it actually suffices to work with rings of positive characteristic, using standard "reduction mod p" techniques. For instance, to show that

$$R = \frac{\mathbb{C}[x,y,z]}{(x^3 - 5y^2 + 7z^3)}$$

has USTP it suffices to show that its reductions mod p,

$$R_p = \underline{\hspace{6cm}}$$

have USTP for all $p \gg 0$. In general, there's a rich theory saying that many properties of a ring in characteristic 0 can be checked mod p sufficiently large. See [HH99, Chapter 2] for details.

12. **Exercise:** How would one define the reduction of a ring such as

$$S = \frac{\mathbb{C}[x,y,z]}{\left(\sqrt{2}x^3 - \pi y^2 + \frac{i}{7}z^3\right)}$$

modulo p?

13. **Now let R have characteristic $p > 0$.** Consider the R-module, $R^{1/p}$ defined by $R^{1/p} :=$ $\underline{\hspace{6cm}}$

Key idea: We can learn a lot about R by studying the R-module structure of R^{1/p^e} for $e > 0$

Note that R^{1/p^e} is always a finitely generated module in our setting.

14. For instance, a theorem of Kunz says that R is regular if and only if R^{1/p^e} is a flat R-module for some (all) $e > 0$ [Kun69].

15. Example from number theory: these modules can be used to detect whether an elliptic curve in positive characteristic is "ordinary" or "supersingular" [BS15].

16. Recall: our goal is to weaken the regularity hypothesis in Theorem 1. The crux of Ein–Lazarsfeld–Smith's proof[1] is the following chain of containments:

$$\mathfrak{p}^{(hn)} \subseteq \sum_{e>0} \sum_{\varphi: R^{1/p^e} \to R} \varphi\left((\mathfrak{p}^{(hn)})^{1/p^e}\right) \subseteq \sum_{e>0} \sum_{\varphi: R^{1/p^e} \to R} \varphi\left(\left(\mathfrak{p}^{(hn)}\right)^{\lfloor p^e/n \rfloor / p^e}\right)^n \subseteq \mathfrak{p}^n$$

The second containment breaks if R is not regular! So we make the sum on the left a little smaller:

Theorem 2 ([Smo18]). *Let R be a normal domain finitely generated over a perfect field k of characteristic p. Then, for all ideals $\mathfrak{a}$ of R, we have*[2]

$$\sum_{e>0} \sum_{\varphi \in \mathscr{D}_e^{(n)}(R)} \varphi\left(\mathfrak{a}^{1/p^e}\right) \subseteq \sum_{e>0} \sum_{\varphi:\, R^{1/p^e} \to R} \varphi\left(\mathfrak{a}^{\lfloor p^e/n \rfloor / p^e}\right)^n,$$

[1] At least, the positive-characteristic analog of their proof. The original proof uses *multiplier ideals* which are, fascinatingly, a close analog of these test ideals that works in characteristic 0. Constructing multiplier ideals requires resolution of singularities, which is not known in positive characteristic.

[2] For the experts: I'm sacrificing precision for clarity by omitting test elements in the sums below.

where $\mathscr{D}_e^{(n)}(R) \subseteq \operatorname{Hom}_R(R^{1/p^e}, R)$ *is the set of maps admitting a lifting to the n-fold tensor product:*

$$\begin{array}{ccc} (R^{\otimes_k n})^{1/p^e} & \dashrightarrow & R^{\otimes_k n} \\ \downarrow & & \downarrow \\ R^{1/p^e} & \xrightarrow{\varphi} & R \end{array}$$

17. I won't explain how this works in this talk, but here's the key take-away I want you to have from this discussion:

    Key idea: This set of maps $\mathscr{D}_e^{(n)}(R)$ is a correction term that accounts for our ring R not being regular. If the correction term is not too bad, then the conclusion of Theorem 1 still holds. [CS18, Theorem 4.1]

18. **Definition:** If $\mathscr{D}_e^{(n)}(R)$ is big enough for the argument to work (for some e), then R is called n-Diagonally F-Regular (n-DFR). If this is true for all $n > 0$, we say R is Diagonally F-Regular (DFR).

19. **Aside for experts:** Concretely, we need the test ideal of $\mathscr{D}^{(n)}(R)$ to be all of R, i.e.

$$\sum_e \sum_{\varphi \in \mathscr{D}_e^{(n)}} \varphi\left(c^{1/p^e}\right) = R$$

    where $c \in R$ is some element such that R_c is regular.

20. So if R is n-DFR, then ____________________ for all $\mathfrak{p}$ of height h.

    The question becomes: Which rings are DFR?

21. Facts about Diagonal F-regularity: regular rings are DFR (exercise! Follows from Kunz's theorem), Segre products of polynomial rings are DFR [CS18] ("non-effective" USTP was known prior to this), tensor products of DFR k-algebras are DFR [CS18] (new rings with USTP!). DFR rings are strongly F-regular. DFR rings are not always Gorenstein and can have arbitrarily small F-signature.

22. **Exercise (hard)**: if $\mathfrak{p}$ is a height 1 prime and torsion element of the divisor class group, then $\mathfrak{p}^{(n)} \neq \mathfrak{p}^n$ for $n \gg 0$. So DFR rings have torsion free divisor class groups [CS18].

4 Diagonal F-regularity of Hibi rings

23. A Hibi ring is a kind of (toric) ring associated to a finite partially ordered set.

24. **Definition** Let $P = \{v_1, \ldots, v_n\}$ be a poset. The associated Hibi ring, $k[P] \subseteq k[x_0, \ldots, x_n]$ is defined as follows: we let $\overline{P} = P \cup \{v_0\}$ where $v_0 \leqslant v_i$ for all i. Then:

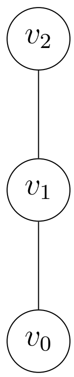
$$k[P] := k\Big[x_0^{a_0} \cdots x_n^{a_n} \Big| \underline{\hspace{4cm}} \Big]$$

25. If you know about poset ideals, then we can also write

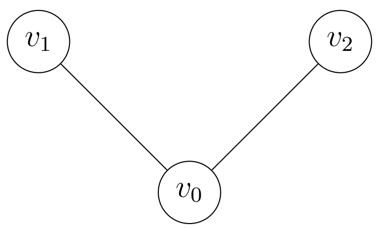
$$k[P] = \frac{k\left[x_I \mid I \subseteq \overline{P} \text{ a poset ideal}\right]}{x_I x_J - x_{I \cup J} x_{I \cap J}}$$

26. We usually denote posets by *Hasse diagrams*: nodes represent elements of $\overline{P}$. Bigger elements are written above smaller elements. Draw an edge between two distinct nodes v_i and v_j if there's no node between them, i.e. if $v_i \leqslant v_k \leqslant v_j$ implies $v_k = v_i$ or $v_k = v_j$. In this case, we say v_j *covers* v_i.

27. Some examples/exercises:

$\overline{P} =$ 

$k[P] =?$

$\overline{Q} =$ 

$k[Q] =?$

28. Checking whether a Hibi ring is n-DFR boils down to solving a complicated combinatorial problem:

    Theorem 3 ([PST18]). *For each i, let r_i be the length of the longest chain going up from v_i in P. Then $k[P]$ is n-DFR if and only if there exists some e such that the following holds: for $0 \leqslant i \leqslant d$ and $1 \leqslant m \leqslant n$, let $\alpha_{i,m}$ be integers in $[0, p^e - 1]$ such that $\sum_{m=1}^{n} \alpha_{j,m} \equiv r_j \pmod{p^e}$ for all j. Set $N_j = \lfloor \sum_{m=1}^{n} \frac{\alpha_{j,m}}{p^e} \rfloor$. For all i, j, and m, let $\varepsilon_{j,i,m} = 1$ if $\alpha_{j,m} > \alpha_{i,m}$ and let $\varepsilon_{j,i,m} = 0$ otherwise. Then there exist $\delta_{i,m} \in \mathbb{Z}$ with*

    (a) $\delta_{i,m} \geqslant 0$ for all m whenever v_i is maximal in P,

    (b) $\delta_{j,m} \leqslant \varepsilon_{j,i,m} + \delta_{i,m}$ for all m whenever v_j covers v_i, and

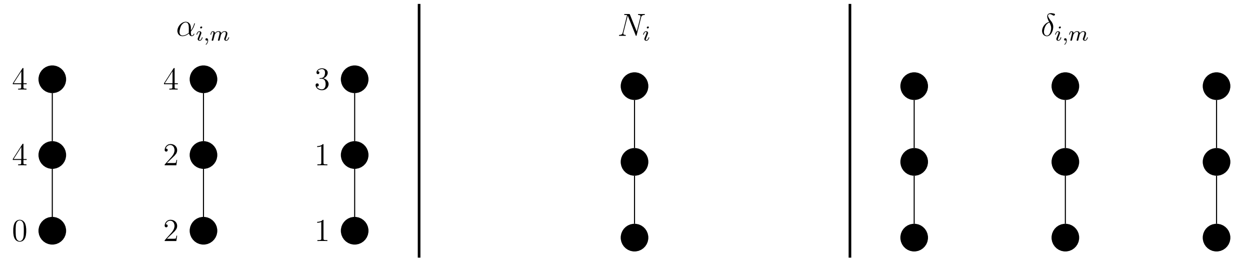
    (c) $\sum_{m=1}^{n} \delta_{j,m} = N_j$

29. **Aside for experts:** The point is that solving this combinatorial problem is the same as constructing a lifting $(R^{\otimes n})^{1/p^e} \to R^{\otimes n}$ of a map $R^{1/p^e} \to R$ that sends $z = x_0^{r_0} \cdots x_n^{r_n}$ to 1. Note that $z \in R$ and R_z is regular.

30. Using this combinatorial description, we were able to show:

    Theorem 4 ([PST18]). *If $k[P]$ is DFR, so is $k[P \cup \{v'\}]$, where v' covers a single element of P.*

31. Example: Checking if $\mathbb{F}_5[x, y, z]$ is 3-DFR:



32. Recall: polynomial rings are DFR. Using theorem 4, which posets (Hasse diagrams) do we know to correspond to DFR Hibi rings?

33. Recall: tensor products of DFR rings are DFR. Here's what the tensor product of two Hibi rings looks like:

34. **Exercise:** Convince yourself you get isomorphic rings doing the tensor product in either order!

35. **Exercise:** What are all the Hibi rings known to be DFR, using Theorem 4 and results about DFR rings in item 20?

36. **Definition:** A *top node* in a poset is a node that covers more than one element. They look like hats in the Hasse diagram.

    Theorem 5 ([PST18])**.** *The Hibi ring $k[P]$ is DFR whenever the set of top nodes of P is* ____

37. The converse to this theorem is not known! Here's the first poset with incomparable top nodes:

38. We know it's 2-DFR (in fact, all Hibi rings are 2-DFR). Dylan Johnson has shown it's 3-DFR.

5 Questions I would like to know the answer to

39. Is the diagonal F-regularity of a toric ring independent of characteristic?

40. Is $\mathscr{D}^{(n)}(R)$ a good metric for the singularities of R? For instance, if $\mathscr{D}_e^{(2)}(R) = \operatorname{Hom}_R(R^{1/p^e}, R)$ for all e, does that imply R is regular? This is true for toric $\mathbb{Q}$-Gorenstein R.

41. Do we always have $\mathscr{D}_e^{(n)}(R) \supseteq \mathscr{D}_e^{(n+1)}(R)$? This is true for toric R.

42. Are rings with large F-signature (say, $> 1/2$) always DFR? Note that such rings have torsion-free divisor class groups by Carvajal-Rojas.

43. What kind of USTP statements can we get if the F-signature of $\mathscr{D}^{(n)}$ is large but < 1?

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