# **Subadditivity and Symbolic Powers**

JMM 2018: AMS Special Session on Commutative Algebra in All Characteristics

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- Answer [ELS '01, HH '02, Hara '05, MS '17]: If R is regular, then h = dim R works for all p!
- Question: can we find a uniform h that works for non-regular rings?

Sketch of ELS/Hara/MS proof: set  $d = \dim R$ .

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Key idea: replace  $\tau(\mathfrak{p}^{(dn)})$  with an ideal so that (2) holds always, and hope (1) holds sometimes

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• An *R*-linear map  $\varphi: F_*^e R \to R$  satisfies

$$\varphi(F_*^e(a+b)) = \varphi(F_*^e a) + \varphi(F_*^e b),$$
  
$$\varphi(F_*^e r^{p^e} x) = \varphi(rF_*^e x) = r\varphi(F_*^e x)$$

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  - " $J, \varphi$  are compatible"

Multiplying a Cart. Alg. by an ideal

• Given  $\mathcal{D}$ ,  $\mathfrak{a}_i \subseteq R$ , construct

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•  $\tau(R, \mathfrak{a}_1 \cdots \mathfrak{a}_n) := \tau(R, C_R \mathfrak{a}_1 \cdots \mathfrak{a}_n)$ 

• Subadditivity: if R regular, then  $\tau(R, \mathfrak{ab}) \subseteq \tau(R, \mathfrak{a})\tau(R, \mathfrak{b})$ .

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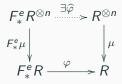
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- For any  $n \in \mathbb{N}$ , define  $\mathscr{D}(n)_e$  as the set of  $\varphi : F_*^e R \to R$  such that



Theorem (S.)

$$\tau(R, \mathcal{D}(n)\mathfrak{a}_1 \cdots \mathfrak{a}_n) \subseteq \tau(R, \mathfrak{a}_1) \cdots \tau(R, \mathfrak{a}_n)$$

Proof.

#### Theorem (S.)

$$\tau(R, \mathcal{D}(n)\mathfrak{a}_1\cdots\mathfrak{a}_n)\subseteq \tau(R,\mathfrak{a}_1)\cdots\tau(R,\mathfrak{a}_n)$$

#### Proof.

$$\operatorname{\mathsf{Hom}}_{R^{\otimes n}}(F_*^eR^{\otimes n},R^{\otimes n})\cong \operatorname{\mathsf{Hom}}_R(F_*^eR,R)^{\otimes n}$$

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The set  $\mathcal{D}(n)$  is constructed specifically so that

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Let k be a field of characteristic p. Then the Segre product  $k[x_0, \ldots, x_r] \# k[y_0, \ldots, y_s]$  is diagonally F-regular.

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Thus,  $\mathfrak{p}^{((r+s+1)n)} \subseteq \mathfrak{p}$  for all  $\mathfrak{p} \in \operatorname{Spec}(k[x_0,\ldots,x_r]\#k[y_0,\ldots,y_s])$ 

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#### Theorem (S.)

If R affine toric, then  $\mathcal{D}(2)_e$  is generated by

$$\{\pi_a \mid P_R \cap (a - P_R) \text{ is "big"}\}$$

$$\mathcal{Z}\subseteq\mathbb{R}^d$$
 is big if  $\forall v\inrac{1}{
ho^e}\mathbb{Z}^d\ \exists s\in\mathbb{Z}:v+s\in\mathcal{Z}$ 

#### **Example**

$$R = k[x, y, u, v]/(xy - uv) \cong k[x, y, u, xyu^{-1}]$$

