On volume and surface area of parallel sets

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Outline

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Parallel sets and their boundaries

Let $A \subset \mathbb{R}^d$ be bounded and $r > 0$. $d_A(x) := \inf_{a \in A} |a - x|$

Closed and open r-parallel set of A:

$$A_r = \{z \in \mathbb{R}^d : d_A(z) \leq r\}, \quad A_{<r} = \{z \in \mathbb{R}^d : d_A(z) < r\}.$$

- $\partial_+ A_r \subseteq \partial A_r \subseteq \partial A_{<r}$

positive boundary:

$$\partial_+ X := \{x \in \partial X : \exists y \notin X \text{ with } d_X(y) = |y - x|\}$$

- $\mathcal{H}^{d-1}(\partial_+ A_r) \leq \mathcal{H}^{d-1}(\partial A_r) \leq \mathcal{H}^{d-1}(\partial A_{<r})$

Volume and surface area

$V_A(r) := \mathcal{H}^d(A_r)$ ... volume of A_r

- continuous and strictly increasing
- *Kneser function*: For $b \geq a > 0$ and $\lambda \geq 1$, [Kneser 51]

$$V_A(\lambda b) - V_A(\lambda a) \leq \lambda^d \left(V_A(b) - V_A(a)\right).$$

- $(V_A)'(r)$ exists up to countably many $r > 0$ [Stacho 76]

$$V_A(r) = \int_0^r (V_A)'(t)dt$$

- $(V_A)'_-(r)$, $(V_A)'_+(r)$ exist for $r > 0$ and $(V_A)'_-(r) \geq (V_A)'_+(r)$
- $\mathcal{M}^{d-1}(\partial A_{<r}) = \dfrac{(V_A)'_-(r) + (V_A)'_+(r)}{2}$

  $(d-1)$-dim. Minkowski content: $\mathcal{M}^{d-1}(B) := \lim_{r \to 0} \dfrac{V_B(r)}{2r}$

- $\mathcal{H}^{d-1}(\partial_+ A_r) = (V_A)'_+(r)$, $\quad r > 0$, [Hug, Last, Weil 01]

Rectifiability of the boundary

$A \subset \mathbb{R}^d$ is k-rectifiable if A is a Lipschitz image of a bounded subset of $\mathbb{R}^k$.

Proposition: For $A \subseteq \mathbb{R}^d$ bounded and any $r > 0$,
$\partial A_{<r}$ and ∂A_r are $(d-1)$-rectifiable.

Consequences: For any $r > 0$,

- $\mathcal{M}^{d-1}(\partial A_r) = \mathcal{H}^{d-1}(\partial A_r)$ and $\mathcal{M}^{d-1}(\partial A_{<r}) = \mathcal{H}^{d-1}(\partial A_{<r})$

For all $r > 0$ except countably many:

$$
\begin{aligned}
V_A'(r) &= \mathcal{M}^{d-1}(\partial A_{<r}) \\
&= \mathcal{H}^{d-1}(\partial A_{<r}) \geq \mathcal{H}^{d-1}(\partial A_r) \geq \mathcal{H}^{d-1}(\partial_+ A_r) \\
&= (V_A)'_+(r) = V_A'(r)
\end{aligned}
$$

- $\mathcal{H}^{d-1}(\partial A_{<r}) = \mathcal{H}^{d-1}(\partial A_r) = \mathcal{H}^{d-1}(\partial_+ A_r) = (V_A)'(r)$

Minkowski content and Minkowski dimension

Let $A \subseteq \mathbb{R}^d$ be compact and $0 \leq s \leq d$.

- s-dimensional Minkowski content of A:

$$\mathcal{M}^s(A) := \lim_{r \to 0} \frac{V_A(r)}{\kappa_{d-s} r^{d-s}} \qquad \left(\kappa_t = \frac{\pi^{t/2}}{\Gamma(t/2+1)}\right).$$

- $\overline{\mathcal{M}}^s(A), \underline{\mathcal{M}}^s(A)$ ... upper and lower Minkowski content
- upper and lower Minkowski dimension:

$$\begin{aligned}
\overline{\dim}_M A &:= \inf\{s : \overline{\mathcal{M}}^s(A) = 0\} = \sup\{s : \overline{\mathcal{M}}^s(A) = \infty\} \\
\underline{\dim}_M A &:= \inf\{s : \underline{\mathcal{M}}^s(A) = 0\} = \sup\{s : \underline{\mathcal{M}}^s(A) = \infty\}
\end{aligned}$$

- $\underline{\dim}_M A \leq \overline{\dim}_M A \leq d$

Surface area based content and dimension

Let $A \subset \mathbb{R}^d$ be compact and $0 \leq s < d$.

- s-dimensional S-content of A:

$$\mathcal{S}^s(A) := \lim_{r\to 0} \frac{\mathcal{H}^{d-1}(\partial A_r)}{(d-s)\kappa_{d-s} r^{d-1-s}} \qquad \mathcal{C}_k^s(A) := \lim_{r\to 0} \frac{C_k(A_r)}{c_{s,k} r^{d-k-s}}$$

- $\overline{\mathcal{S}}^s(A), \underline{\mathcal{S}}^s(A)$ ... upper and lower S-content
- $s = d$: $\lim\limits_{r\to 0} \dfrac{\mathcal{H}^{d-1}(\partial A_r)}{r^{-1}} = 0 \Longrightarrow$ Set $\mathcal{S}^d(A) := 0$.
- upper and lower S-dimension:

$$\begin{aligned} \overline{\dim}_S A &:= \inf\{s : \overline{\mathcal{S}}^s(A) = 0\} = \sup\{s : \overline{\mathcal{S}}^s(A) = \infty\} \\ \underline{\dim}_S A &:= \inf\{s : \underline{\mathcal{S}}^s(A) = 0\} = \sup\{s : \underline{\mathcal{S}}^s(A) = \infty\} \end{aligned}$$

- $\underline{\dim}_S A \leq \overline{\dim}_S A \leq d$

Relations between S-content and Minkowski content

Theorem: Let $A \subset \mathbb{R}^d$ be compact with $V_A(0) = 0$.
For $0 \leq s \leq d$,

$$\underline{\mathcal{S}}^s(A) \leq \underline{\mathcal{M}}^s(A) \leq \overline{\mathcal{M}}^s(A) \leq \overline{\mathcal{S}}^s(A)$$

- If $\mathcal{S}^s(A)$ exists, then $\mathcal{M}^s(A)$ exists and $\mathcal{M}^s(A) = \mathcal{S}^s(A)$.
- $\underline{\dim}_S A \leq \underline{\dim}_M A \leq \overline{\dim}_M A \leq \overline{\dim}_S A$

Idea of proof: variation of l'Hôpitals rule

$$\mathcal{M}^s(A) := \lim_{r \to 0} \frac{V_A(r)}{\kappa_{d-s} r^{d-s}} = \lim_{r \to 0} \frac{(V_A)'(r)}{(d-s)\kappa_{d-s} r^{d-s-1}} = \mathcal{S}^s(A).$$

Relations between S-content and Minkowski content

Let $A \subset \mathbb{R}^d$ be compact with $V_A(0) = 0$ and $0 \leq s \leq d$.

from [Stacho] and the integral representation of V_A:

- $\overline{\mathcal{M}}^s(A) \leq \overline{\mathcal{S}}^s(A) \leq \frac{d}{d-s}\overline{\mathcal{M}}^s(A)$
- $\overline{\dim}_M A = \overline{\dim}_S A$

from isoperimetric inequality:

- $c\left(\underline{\mathcal{M}}^{s\frac{d}{d-1}}(A)\right)^{\frac{d-1}{d}} \leq \underline{\mathcal{S}}^s(A) \leq \underline{\mathcal{M}}^s(A)$
- $\frac{d-1}{d}\underline{\dim}_M A \leq \underline{\dim}_S A \leq \underline{\dim}_M A$

All inequalities can be strict!

$\frac{d-1}{d}$ is best possible!

S-dimension of self-similar sets

- fractal curvatures:

$$\mathcal{C}_k(F) = \lim_{r\to 0} \frac{C_k(F_r)}{r^{d-k-s_k}} \text{ with } s_k := \inf\{t : \lim_{r\to 0} \frac{C_k(F_r)}{r^{d-k-t}} = 0\}$$

- $k = d-1$: $s_{d-1} = \overline{\dim}_S F$, $\mathcal{C}_{d-1}(F) = c\mathcal{S}^{s_{d-1}}(F)$
- known [W. '06]: If F satisfies OSC and has polyconvex F_r, then

$$s_k \leq dim_M F$$

  (and some formula to check whether '=' holds)
- **Theorem:** If F satisfies OSC and $dim_M F < d$, then

$$s_{d-1} = \overline{\dim}_S F = \dim_M F$$

- $\Longrightarrow 0 < \overline{\mathcal{M}}^D(F) \leq \overline{\mathcal{S}}^D(F) \leq \frac{d}{d-1}\overline{\mathcal{M}}^D(F) < \infty$

Existence of S-content

average S-content of A:

$$\widetilde{\mathcal{S}}^s(A) = \lim_{t\to 0} \frac{1}{|\log t|} \int_t^1 \frac{\mathcal{H}^{d-1}(\partial A_r)}{(d-s)\kappa_{d-s} r^{d-1-s}} \, d\log r$$

Theorem: Suppose F satisfies OSC and $D < d$.

(i) $\widetilde{\mathcal{S}}^D(F)$ exists

(ii) If F is non-lattice, then $\mathcal{S}^D(F)$ exists.

F non-lattice $\Leftrightarrow$ $\{\ln r_1, \dots, \ln r_N\}$ are not rationally dependent

Corollary:

- If F is non-lattice, then $\mathcal{S}^D(F) = \mathcal{M}^D(F)$.
- $\widetilde{\mathcal{S}}^D(F) = \widetilde{\mathcal{M}}^D(F)$

Fractal strings - measurability

$F \subset \mathbb{R}$ compact, $\dim_M F = D \in (0,1)$,
$\mathcal{L} = (l_j)_{j=1}^{\infty}$ associated fractal string

Theorem: The following assertions are equivalent:

(i) $0 < \underline{\mathcal{M}}^D(F) \leq \overline{\mathcal{M}}^D(F) < \infty$

(ii) $0 < \underline{\mathcal{S}}^D(F) \leq \overline{\mathcal{S}}^D(F) < \infty$

(iii) $l_j \approx j^{-1/D}$ as $j \to \infty$

Theorem: (Minkowski measurability)
The following assertions are equivalent:

(i) F is Minkowski measurable

(ii) F is S-measurable, i.e., $0 < \mathcal{S}^D(F) < \infty$

(iii) $l_j \sim L j^{-1/D}$ as $j \to \infty$ for some $L > 0$.

Fractal strings - the sound

$\Omega \subset \mathbb{R}$ bdd. domain, $F = \partial\Omega$, $\mathcal{L} = (l_j)_{j=1}^{\infty}$ assoc. fractal string

Eigenvalue counting function (of Dirichlet Laplacian Δ on Ω):

$$N(\lambda) := \#\{j \in \mathbb{N} : \lambda_j < \lambda\}, \lambda > 0$$

where $(\lambda_i)_{i\in\mathbb{N}}$ are the eigenvalues of Δ.

Recall: $$N(\lambda) = \sum_{i=1}^{\infty} [l_j x] \qquad \text{with } x = \sqrt{\lambda}/\pi$$

Theorem: (Weyl-Berry-Conjecture) [Lapidus & Pomerance'92]
If $F = \partial\Omega$ is Minkowski measurable with $\dim_M F = D \in (0,1)$ then

$$N(\lambda) = \phi(\lambda) - c_D \mathcal{M}^D(F)\lambda^{D/2} + o(\lambda^{D/2}), \qquad \text{as } \lambda \to \infty.$$

To understand the second term, study the asymptotics of

$$\phi(\lambda) - N(\lambda) = \sum_{j=1}^{\infty} l_j x - \sum_{j=1}^{\infty} [l_j x] = \sum_{j=1}^{\infty} \{l_j x\} =: \delta(x)$$

$[x]$ integer part, $\{x\}$ fractional part of x

Fractal strings - the sound

$F \subset \mathbb{R}$ compact, $\dim_M F = D \in (0,1)$,
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Theorem: The following assertions are equivalent:

(i) $0 < \underline{\mathcal{M}}^D(F) \leq \overline{\mathcal{M}}^D(F) < \infty$

(ii) $0 < \underline{\mathcal{S}}^D(F) \leq \overline{\mathcal{S}}^D(F) < \infty$

(iii) $l_j \approx j^{-1/D}$ as $j \to \infty$

(iv) $\delta(x) \approx x^D$ as $x \to \infty$

Theorem: (Minkowski measurability)
The following assertions are equivalent:

(i) F is Minkowski measurable

(ii) F is S-measurable, i.e., $0 < \mathcal{S}^D(F) < \infty$

(iii) $l_j \sim L j^{-1/D}$ as $j \to \infty$ for some $L > 0$.

(iv) $\delta(x) \sim c_D x^D$ as $x \to \infty$

Fractal strings - upper bounds

What happens if $\underline{\dim}_M F < \overline{\dim}_M F$?

Theorem: (One sided upper bound)
The following assertions are equivalent:

(i) $\overline{\mathcal{M}}^D(F) < \infty$

(ii) $\overline{\mathcal{S}}^D(F) < \infty$

(iii) $l_j = O(j^{-1/D})$ as $j \to \infty$

(iv) $\delta(x) = O(x^D)$ as $x \to \infty$

More precisely, if $\overline{\dim}_M F = D \in (0,1)$ and

$$\overline{\delta}^D(\mathcal{L}) := \limsup_{x\to\infty} x^{-D}\delta(x)$$

then

$$c_1\overline{\mathcal{M}}^D(F) \le c_1\overline{\mathcal{S}}^D(F) \le \overline{\delta}^D(\mathcal{L}) \le c_2\overline{\mathcal{M}}^D(F)$$

Fractal strings - lower bounds

One sided lower bounds?

$$\underline{\mathcal{M}}^D(F)?, \underline{\mathcal{S}}^D(F)?, \underline{\mathcal{M}}^D(F) - \underline{\mathcal{S}}^D(F)?$$

For $D \in (0,1)$, let

$$\underline{\delta}^D(\mathcal{L}) := \liminf_{x \to \infty} x^{-D}\delta(x).$$

Observation: Let $D \in (0,1)$. Then

$$c_1 \liminf_{r \searrow 0} \left(\frac{V(F_r)}{r^{D-1}} - \frac{\mathcal{H}^0(\partial F_r)}{r^{-D}} \right) \leq \underline{\delta}^D(\mathcal{L}) \leq c_2 \underline{\mathcal{M}}^D(F).$$

In particular,

$$\text{"}\underline{\dim}_{(M-S)} F \leq \underline{\dim}_\delta F \leq \underline{\dim}_M F\text{"}.$$

Open Questions

- S-measurability = Minkowski measurability in $\mathbb{R}^d$?
- Characterization of $\underline{\dim}_\delta F$
- What is the relation between $\underline{\dim}_S$ and complex dimensions?
- Consequences for tube formulas?
- Behaviour of other fractal curvatures? Representations as higher order derivatives of the volume?

References

- J. Rataj & S. Winter: *On volume and surface area of parallel sets*. Preprint 2009 (arXiv:0905.3279).
- S. Winter: *Sets of small lower S-dimension.* In preparation.