6

1(a) Use integration by parts to evaluate $\int_1^{\ln 5} xe^x \, dx$.

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x dx$

$\int_1^{\ln 5} xe^x dx = xe^x \Big|_1^{\ln 5} - \int_1^{\ln 5} e^x dx$

$= xe^x - e^x \Big|_1^{\ln 5}$

$= \ln 5 \cdot e^{\ln 5} - e^{\ln 5} - e^1 + e^1$

$= \boxed{(\ln 5 - 1) \cdot 5}$

6

(b) Use a substitution to evaluate $\int_1^{e^4} \frac{\ln x}{x} \, dx$.

$u = \ln x \quad , \quad du = \frac{1}{x} dx$

$\int_1^{e^4} \frac{\ln x}{x} dx = \int_0^4 u \, du$

$= \frac{1}{2} u^2 \Big|_0^4$

$= \boxed{8}$

6

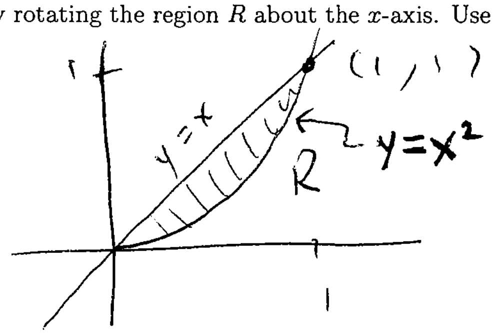
2(a) Write out the form of the partial fraction decomposition for the rational function $\frac{x^5 - x^2}{(x-1)(x^2+1)(x^4-1)}$.
(Do not solve for the constants or proceed any further.)

$$Q(x) = (x-1)(x^2+1)(x^4-1)$$
$$= (x-1)(x^2+1)(x^2+1)(x^2-1)$$
$$= (x-1)(x^2+1)^2(x+1)(x-1)$$
$$= (x-1)^2(x^2+1)^2(x+1)$$

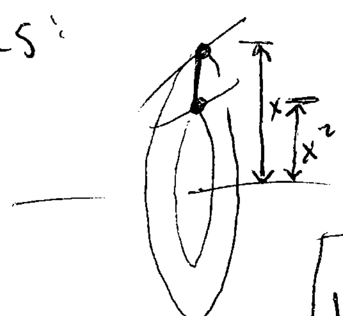
$$\boxed{\frac{P(x)}{Q(x)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} + \frac{G}{x+1}}$$

7

2(b) Let R be the region between the curves $y = x$ and $y = x^2$. Set up an integral whose value equals the volume of the solid obtained by rotating the region R about the x-axis. Use shells or washers, as you prefer. (Do not evaluate the integral.)



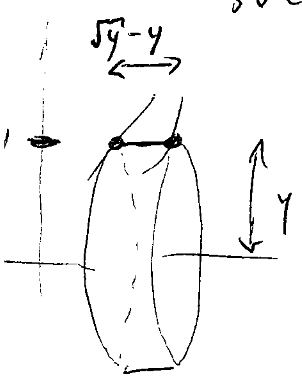
washers:



$$A(x) = \pi x^2 - \pi(x^2)^2$$
$$= \pi x^2(1-x^2)$$

$$\boxed{V = \int_0^1 \pi x^2(1-x^2)\,dx}$$

shells:



$$A(y) = 2\pi y(\sqrt{y} - y)$$

$$\boxed{V = \int_0^1 2\pi y(\sqrt{y} - y)\,dy}$$

3. Use the trigonometric substitution $x = 2\tan\theta$ to evaluate the integral $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$ as follows:

5
5
3

(a) Transform into an integral involving θ
(b) Evaluate the integral from (a)
(c) Use a right triangle to express the result in terms of x

$$x = 2\tan\theta, \quad dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{(2\tan\theta)^3}{\sqrt{(2\tan\theta)^2+4}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{16\tan^3\theta\sec^2\theta}{\sqrt{4\sec^2\theta}}\,d\theta = \int \frac{16\tan^3\theta\sec^2\theta}{2\sec\theta}\,d\theta$$

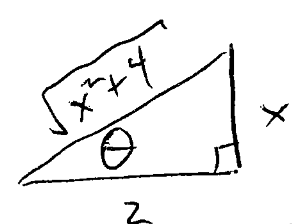
$$= \boxed{8\int \tan^3\theta\sec\theta\, d\theta} = 8\int \tan^2\theta\sec\theta\tan\theta\, d\theta$$

$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta$$

$$u = \sec\theta, \quad du = \sec\theta\tan\theta\, d\theta$$

$$= 8\int (u^2-1)\,du = 8\left(\tfrac{1}{3}u^3 - u\right) + C$$

$$= \boxed{\frac{8}{3}\sec^3\theta - 8\sec\theta + C}$$



$$\Rightarrow \sec\theta = \frac{\sqrt{x^2+4}}{2}$$

$$\frac{8}{3}\,\frac{(x^2+4)^{3/2}}{8} - 4\sqrt{x^2+4} + C$$

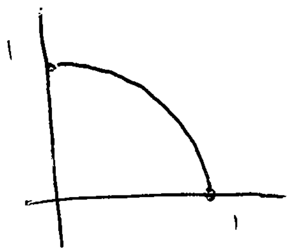
$$= \boxed{\frac{1}{3}(x^2+4)^{3/2} - 4(x^2+4)^{1/2} + C}$$

2
7

4(a) Let C be the curve which is the part of the unit circle in the first quadrant.

(i) Write C as the graph of a function $y = f(x)$, $0 \le x \le 1$.

(ii) It can be shown easily that $ds = \dfrac{1}{\sqrt{1-x^2}}\,dx$. Compute the length of C by evaluating an integral. If the integral is improper, show the details of how you handle it.



$$x^2+y^2=1 \Rightarrow \boxed{y=\sqrt{1-x^2}}$$

$$\text{Length} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

Improper; undefined at $x=1$.

$$L = \lim_{t\to1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to1^-} \sin^{-1}x \Big|_0^t$$

$$= \lim_{t\to1^-}\left(\sin^{-1}t - \sin^{-1}0\right) = \sin^{-1}1 - \sin^{-1}0$$

(since $\sin^{-1}x$ is defined and cont. at $x=1$)

$$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

5

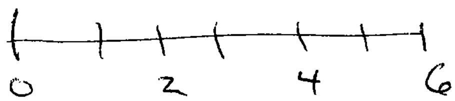
(b) What are the domain and range for $f(x) = \ln(\tan^{-1}(x))$?

Domain : $(0, \infty)$

Range : $(-\infty, \ln \pi/2)$

4

5(a) Carefully write out all the terms of the Riemann Sum for the function $f(x) = x^2$ over the interval $[0, 6]$, using three sub-intervals and right endpoints. (Draw a picture!)



$\Delta x = 2$

$$\boxed{2\left((2)^2 + (4)^2 + (6)^2\right)}$$

5

5(b) Find the limits:

(i) $\displaystyle\lim_{x\to-\infty} x^2 e^x$ $\displaystyle = \lim_{x\to-\infty} \frac{x^2}{e^{-x}}$ indeterminate, type $\frac{\infty}{\infty}$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$ indeterminate, type $\frac{\infty}{\infty}$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = \boxed{0}$$ since denominator $\to +\infty$.

5

(ii) $\displaystyle\lim_{t\to 0} \frac{5^t - 3^t}{t}$ indeterminate, type $\frac{0}{0}$

since $5^0 - 3^0 = 0$

$$\overset{L'H}{=} \lim_{t\to 0} \frac{(\ln 5)5^t - (\ln 3)3^t}{1} = \ln 5 - \ln 3$$

$$= \boxed{\ln\left(5/3\right)}$$

5 5 4

6. This problem concerns the curve given by $y = 3 + \frac{1}{2}\cosh(2x)$, from $x = 0$ to $x = 1$.

(a) Find ds for this curve. (An indentity may be helpful.)

(b) Calculate the length of the curve.

(c) Write down an integral whose value equals the surface area produced when the curve is rotated about the x-axis. (Do not evaluate this integral.)

(a) $\frac{dy}{dx} = \frac{1}{2}\sinh(2x)\cdot 2 = \sinh(2x)$

$1+\left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2(2x) = \cosh^2(2x)$

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \boxed{\cosh(2x)\,dx}$

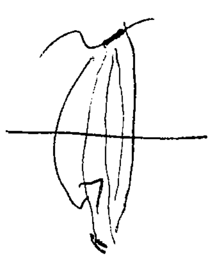
(b) length $= \int_0^1 ds = \int_0^1 \cosh(2x)\,dx$

$= \frac{1}{2}\sinh(2x)\Big|_0^1$

$= \frac{1}{2}(\sinh(2) - \underbrace{\sinh(0)}_{0})$

$= \boxed{\frac{e^2 - e^{-2}}{4}}$

(c)



$SA = \int_0^1 2\pi y\, ds$

$= \boxed{\int_0^1 2\pi\left(3 + \frac{1}{2}\cosh(2x)\right)\cosh(2x)\,dx}$

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\sin(2x) = 2\sin x \cos x$

$\cos(2x) \;=\; \cos^2 x - \sin^2 x \;=\; 2\cos^2 x - 1 \;=\; 1 - 2\sin^2 x$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

+2

Extra Credit Find the derivative of $f(x) = \int_e^{\ln x} \frac{1}{\ln(t)}\, dt$.

$$f'(x) = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln x)$$

$$= \boxed{\frac{1}{x \ln(\ln(x))}}$$