Homework 15

8.1

2. Using the arc length formula with $y = \sqrt{2 - x^2} \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{2 - x^2}}$, we get

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1 + \frac{x^2}{2 - x^2}}\,dx = \int_0^1 \frac{\sqrt{2}\,dx}{\sqrt{2 - x^2}} = \sqrt{2}\int_0^1 \frac{dx}{\sqrt{(\sqrt{2})^2 - x^2}}$$

$$= \sqrt{2}\left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1 = \sqrt{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0\right] = \sqrt{2}\left[\tfrac{\pi}{4} - 0\right] = \sqrt{2}\,\tfrac{\pi}{4}$$

The curve is a one-eighth of a circle with radius $\sqrt{2}$, so the length of the arc is $\frac{1}{8}(2\pi \cdot \sqrt{2}) = \sqrt{2}\,\frac{\pi}{4}$, as above.

10. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2} \;\Rightarrow\; \dfrac{dx}{dy} = \frac{1}{2}y^3 - \frac{1}{2}y^{-3} \;\Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4}y^{-6} = \frac{1}{4}y^6 + \frac{1}{2} + \frac{1}{4}y^{-6} = \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)^2$. So

$L = \int_1^2 \sqrt{\left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)^2}\,dy = \int_1^2 \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)dy = \left[\frac{1}{8}y^4 - \frac{1}{4}y^{-2}\right]_1^2 = \left(2 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right)$

$= 2 + \frac{1}{16} = \frac{33}{16}$.

18. $y = 1 - e^{-x} \;\Rightarrow\; y' = -(-e^{-x}) = e^{-x} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + e^{-2x}$. So

$$L = \int_0^2 \sqrt{1 + e^{-2x}}\,dx = \int_1^{e^{-2}} \sqrt{1 + u^2}\left(-\frac{1}{u}\,du\right) \qquad [u = e^{-x}]$$

$$\overset{23}{=} \left[\ln\left|\frac{1 + \sqrt{1 + u^2}}{u}\right| - \sqrt{1 + u^2}\right]_1^{e^{-2}} \qquad \text{[or substitute } u = \tan\theta\text{]}$$

$$= \ln\left|\frac{1 + \sqrt{1 + e^{-4}}}{e^{-2}}\right| - \sqrt{1 + e^{-4}} - \ln\left|\frac{1 + \sqrt{2}}{1}\right| + \sqrt{2}$$

$$= \ln\left(1 + \sqrt{1 + e^{-4}}\right) - \ln e^{-2} - \sqrt{1 + e^{-4}} - \ln\left(1 + \sqrt{2}\right) + \sqrt{2}$$

$$= \ln\left(1 + \sqrt{1 + e^{-4}}\right) + 2 - \sqrt{1 + e^{-4}} - \ln\left(1 + \sqrt{2}\right) + \sqrt{2}$$

8.2

12. $x = 1 + 2y^2 \;\Rightarrow\; 1 + (dx/dy)^2 = 1 + (4y)^2 = 1 + 16y^2$.

So $S = 2\pi \int_1^2 y\sqrt{1 + 16y^2}\,dy = \frac{\pi}{16}\int_1^2 (16y^2 + 1)^{1/2} 32y\,dy = \frac{\pi}{16}\left[\frac{2}{3}(16y^2 + 1)^{3/2}\right]_1^2 = \frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$.

26. $S = \int_0^\infty 2\pi y\sqrt{1 + (dy/dx)^2}\,dx = 2\pi\int_0^\infty e^{-x}\sqrt{1 + (-e^{-x})^2}\,dx \qquad [y = e^{-x},\ y' = -e^{-x}]$.

Evaluate $I = \int e^{-x}\sqrt{1 + (-e^{-x})^2}\,dx$ by using the substitution $u = -e^{-x}$, $du = e^{-x}\,dx$:

$I = \int \sqrt{1 + u^2}\,du \overset{21}{=} \frac{1}{2}u\sqrt{1 + u^2} + \frac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right) + C = \frac{1}{2}(-e^{-x})\sqrt{1 + e^{-2x}} + \frac{1}{2}\ln\left(-e^{-x} + \sqrt{1 + e^{-2x}}\right) + C.$

Returning to the surface area integral, we have

$$S = 2\pi\lim_{t\to\infty}\int_0^t e^{-x}\sqrt{1 + (-e^{-x})^2}\,dx = 2\pi\lim_{t\to\infty}\left[\tfrac{1}{2}(-e^{-x})\sqrt{1 + e^{-2x}} + \tfrac{1}{2}\ln\left(-e^{-x} + \sqrt{1 + e^{-2x}}\right)\right]_0^t$$

$$= 2\pi\lim_{t\to\infty}\left\{\left[\tfrac{1}{2}(-e^{-t})\sqrt{1 + e^{-2t}} + \tfrac{1}{2}\ln\left(-e^{-t} + \sqrt{1 + e^{-2t}}\right)\right] - \left[\tfrac{1}{2}(-1)\sqrt{1 + 1} + \tfrac{1}{2}\ln\left(-1 + \sqrt{1 + 1}\right)\right]\right\}$$

$$= 2\pi\left\{\left[\tfrac{1}{2}(0)\sqrt{1} + \tfrac{1}{2}\ln(0 + \sqrt{1})\right] - \left[-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(-1 + \sqrt{2}\right)\right]\right\}$$

$$= 2\pi\left\{[0] + \tfrac{1}{2}\left[\sqrt{2} - \ln\left(\sqrt{2} - 1\right)\right]\right\} = \pi\left[\sqrt{2} - \ln\left(\sqrt{2} - 1\right)\right]$$