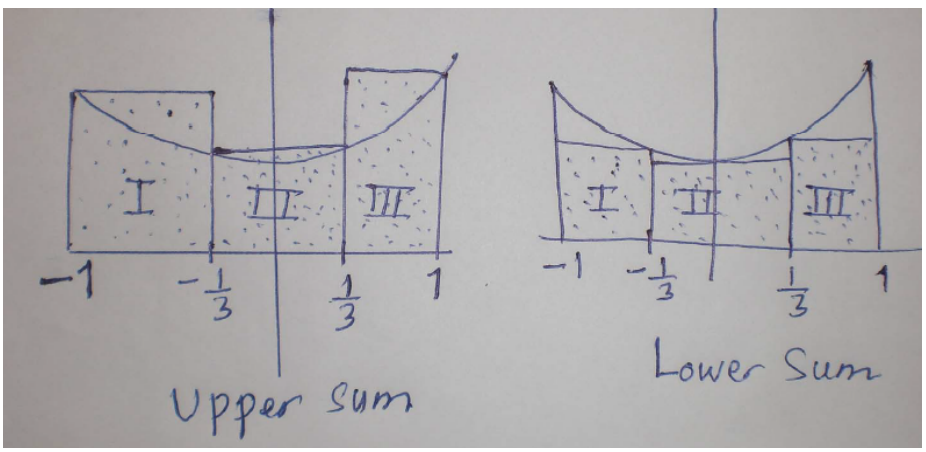
Homework #2 Solutions
Math 2423 Spring 2012

4.1#8 $f(x) = 1 + x^2,\ -1 \le x \le 1$. length of subinterval $= \triangle x = 2/n$. We first find the upper and lower sums by taking $n = 3$. For $n = 3,\ \triangle x = 2/3$. From the figure below (left), upper sum is the sum of the area of rectangles $I,\ II$ and III. Therefore,

$$\begin{aligned} \text{upper sum} &= [f(-1) + f(\frac{1}{3}) + f(1)].\triangle x \\ &= [2 + \frac{10}{9} + 2].\frac{2}{3} = \frac{92}{27}. \end{aligned}$$

The lower sum is (see figure below right),

$$\begin{aligned} \text{lower sum} &= [f(-\frac{1}{3}) + f(0) + f(\frac{1}{3})].\triangle x \\ &= [\frac{10}{9} + 1 + \frac{10}{9}].\frac{2}{3} = \frac{58}{27}. \end{aligned}$$



Next, for $n = 4,\ \triangle x = \frac{2}{4} = \frac{1}{2}$. (You can make a similar diagram for this case). Then

$$\begin{aligned} \text{upper sum} &= [f(-1) + f(-\frac{1}{2}) + f(\frac{1}{2}) + f(1)].\triangle x \\ &= [2 + \frac{5}{4} + \frac{5}{4} + 2].\frac{1}{2} = \frac{13}{4} \end{aligned}$$

and

$$\begin{aligned} \text{lower sum} &= [f(-\frac{1}{2}) + f(0) + f(0) + f(\frac{1}{2})].\triangle x \\ &= [\frac{5}{4} + 1 + 1 + \frac{5}{4}].\frac{1}{2} = \frac{9}{4}. \qquad \blacksquare \end{aligned}$$

4.1#20 Here $f(x) = x^2 + \sqrt{1 + 2x},\ 4 \le x \le 7$. We compute $\triangle x$ and x_i as follows

$$\triangle x = \frac{b-a}{n} = \frac{7-4}{n} = \frac{3}{n}, \text{ and } x_i = a + i\triangle x = 4 + \frac{3i}{n}.$$

Using the limit formula for area, we obtain

$$\begin{aligned}
A &= \lim_{n\longrightarrow\infty} R_n = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} f(x_i)\triangle x \\
&= \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} f\left(4+\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} \left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3i}{n}\right)}\right].\frac{3}{n} \quad \blacksquare
\end{aligned}$$

4.2#6 Here $n = 6$ and $\triangle x = \dfrac{b-a}{2} = \dfrac{4-(-2)}{6} = \dfrac{6}{6} = 1.$

(a) We have to find R_6. Right end-points are $-1, 0, 1, 2, 3, 4$. Therefore

$$\begin{aligned}
\int_{-2}^{4} g(x)dx &\equiv R_6 = [g(-1)+g(0)+g(1)+g(2)+g(3)+g(4)]\triangle x \\
&= [-\frac{3}{2}+0+\frac{3}{2}+\frac{1}{2}+(-1)+\frac{1}{2}].1 \\
&= 0.
\end{aligned}$$

(b) We have to find L_6. Left end-points are $-2, -1, 0, 1, 2, 3$. Therefore

$$\begin{aligned}
\int_{-2}^{4} g(x)dx &\equiv L_6 = [g(-2)+g(-1)+g(0)+g(1)+g(2)+g(3)]\triangle x \\
&= [0+(-\frac{3}{2})+0+\frac{3}{2}+\frac{1}{2}+(-1)].1 \\
&= -\frac{1}{2}.
\end{aligned}$$

(c) We have to find M_6. Mid points are $-3/2, -1/2, 1/2, 3/2, 5/2, 7/2$. Therefore

$$\begin{aligned}
\int_{-2}^{4} g(x)dx &\equiv M_6 = [g(-\frac{3}{2})+g(-\frac{1}{2})+g(\frac{1}{2})+g(\frac{3}{2})+g(\frac{5}{2})+g(\frac{7}{2})]\triangle x \\
&= [-1+(-1)+1+1+0+(-\frac{1}{2})].1 \\
&= -\frac{1}{2} \quad \blacksquare
\end{aligned}$$

4.2#18 We have given the limit

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i}\triangle x \quad \text{on } [\pi, 2\pi].$$

and we have to find the definite integral corresponding to this expression. The idea is to compare this expression with the definition of definite integral,

$$\int_a^b f(x)dx = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} f(x_i) \triangle x.$$

In our case, a is π, and b is 2π. Next we have to find $f(x)$. From the given expression, we have $f(x_i) = \frac{\cos x_i}{x_i}$. This gives $f(x) = \frac{\cos x}{x}$. Therefore the definite integral for the given expression is

$$\int_{\pi}^{2\pi} \frac{\cos x}{x} dx. \qquad \blacksquare$$

4.2#34 (a) $\int_0^2 g(x)dx$ is the area of triangle. This area of triangle is the half of the area of rectangle of height 4 and base 2. Therefore

$$\begin{aligned} \int_0^2 g(x)dx &= \text{area of triangle} \\ &= \frac{1}{2} \times \text{area of rectangle} \\ &= \frac{1}{2} \times (4 \times 2) = 4. \end{aligned}$$

(b) $\int_2^6 g(x)dx$ is the negative of the area of semicircle. The area of semicircle is the half of the area of circle. From the given graph, radius of semicircle $(r) = 2$. Therefore

$$\begin{aligned} \int_2^6 g(x)dx &= \text{negative of the area of semicircle} \\ &= -\frac{1}{2} \times \text{area of circle} \\ &= -\frac{1}{2} \times \pi r^2 \\ &= -\frac{1}{2} \times \pi (2)^2 = -2\pi. \end{aligned}$$

(c) $\int_6^7 g(x)dx$ is the area of triangle. By the similar method what we did in (a), we have

$$\begin{aligned} \int_6^7 g(x)dx &= \text{area of triangle} \\ &= \frac{1}{2} \times \text{area of rectangle} \\ &= \frac{1}{2} \times (1 \times 1) = \frac{1}{2}. \end{aligned}$$

Finally, total area is given by

$$\begin{aligned}
\int_0^7 g(x)dx &= \left(\int_0^2 + \int_2^6 + \int_6^7\right) g(x)\ dx \\
&= \int_0^2 g(x)dx + \int_2^6 g(x)dx + \int_6^7 g(x)dx \\
&= 4 - 2\pi + \frac{1}{2} \\
&= 4 - 2\pi + 0.5 \\
&= 4.5 - 2\pi. \qquad \blacksquare
\end{aligned}$$