Homework #9

6.4

Use logarithm differentiation to find the derivative.

50 $y = \sqrt{x}^x$

$\Rightarrow \ln y = \ln \sqrt{x}^x$

$\Rightarrow \ln y = x \ln \sqrt{x}$

Take the derivative: $\frac{1}{y}\frac{dy}{dx} = \ln\sqrt{x} + x\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln\sqrt{x} + \frac{1}{2}$

$\frac{dy}{dx} = y\left(\ln\sqrt{x} + \frac{1}{2}\right) = \sqrt{x}^x\left(\ln\sqrt{x} + \frac{1}{2}\right)$

Note: $y = \sqrt{x}^x$

76 $\int \frac{\sin(\ln x)}{x}\,dx$ Use substitution. $u = \ln x \quad du = \frac{1}{x}dx$

$= \int \sin(u)\,du = -\cos(u) + C = -\cos(\ln x) + C$

6.5

4 Bacteria count was 400 after 2 hours $\Rightarrow y(2) = 400$
Bacteria count was 25,600 after 6 hours $\Rightarrow y(6) = 25,600$

Since $y(t) = y(0)e^{kt}$, then we have

$y(2) = y(0)e^{2k} = 400$ and $y(6) = y(0)e^{6k} = 25,600$

Two variables and two unknowns.

(a) Solve for k.

You can divide the two equations:

$$\frac{y(0)e^{6k}}{y(0)e^{2k}} = \frac{25600}{400} = 64$$

$$\Rightarrow \quad e^{4k} = 64 \Rightarrow \ln e^{4k} = \ln 64 = \ln 2^6$$

$$\Rightarrow \quad 4k = 6\ln 2 \Rightarrow k = \tfrac{3}{2}\ln 2 \approx 1.04$$

$$\boxed{k \approx 104\%}$$

(b) Intial size? i.e. what is $y(0)$?

$$y(2) = 400 = y(0)\, e^{(\frac{3}{2}\ln 2)\cdot 2} \Rightarrow y(0) \quad \frac{400}{e^{(\frac{3}{2}\ln 2)2}} = \boxed{50}$$

(c) Write a nice expression:

$$\boxed{\begin{array}{l} y(t) = 50e^{\frac{3}{2}\ln 2\cdot t} \text{ or } 50e^{1.04t} \\ \text{or} \\ y(t) = 50\left(e^{\frac{3}{2}\ln 2}\right)^t \approx 50(2.828)^t \end{array}}$$

(d) How many cells after 4.5 hours?

$$y(4.5) = 50\, e^{\frac{3}{2}\ln 2\cdot 4.5} = \boxed{5382 \text{ cells}}$$

(e) Find the rate of growth after 4.5 hours. (Find the derivative)

$$y(t) = 50e^{\frac{3}{2}\ln 2\cdot t}$$
$$y'(t) = 50e^{\frac{3}{2}\ln 2\cdot t} \cdot \tfrac{3}{2}\ln 2$$

So $\boxed{y'(4.5) = 5596 \text{ cells per hour}}$

(f) At what t will $y(t) = 50{,}000$

$$50{,}000 = 50e^{\frac{3}{2}\ln 2\cdot t} \Rightarrow \ln 1000 = \tfrac{3}{2}\ln 2\, t \Rightarrow \boxed{t = 6.643 \text{ hours}}$$

⑧ Strontium-90 has a half-life of 28 days.

(a) With intial mass of 50 mg, how much is left after t days.

Knowing the half-life tells us that

$y(28) = 50e^{k(28)} = 25$

$\Rightarrow e^{28k} = 1/2 \Rightarrow \ln e^{28k} = \ln 1/2 \Rightarrow 28k = -\ln 2$

$\Rightarrow k = \dfrac{-\ln 2}{28} \approx -.0248$

So a nice formula is $\boxed{y(t) = 50e^{-.0248t}}$

(b) After 40 days, how much STRONTIUM-90?

$y(40) = 50e^{(-.0248 \cdot 40)} = \boxed{18.542 \text{ mg}}$

(c) How many days until we only have 2 mg left?
Solve for t when $y(t) = 2$

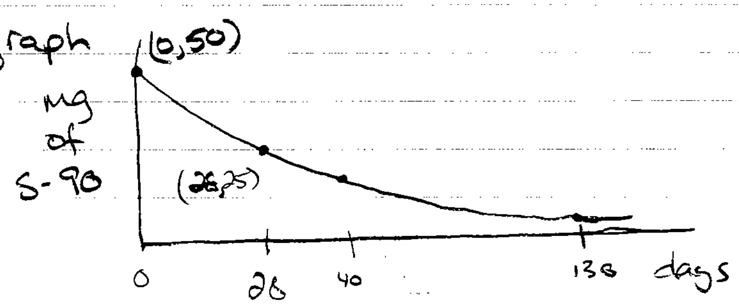
$50e^{(-.0248t)} = 2 \Rightarrow e^{-.0248t} = 1/25$

$\Rightarrow \ln e^{-.0248t} = \ln 1/25$

$\Rightarrow -.0248t = -2\ln 5$

$\Rightarrow \boxed{t = 129.79}$ So it takes $\boxed{130 \text{ days.}}$

(d) Sketch a graph



(20) (a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?

Future Value = (Principal) $e^{(\text{rate})(\text{time})}$

$FV = Pe^{rt}$

Double means $FV = 2P$

$2P = Pe^{.06t} \Rightarrow 2 = e^{.06t} \Rightarrow \ln 2 = .06t$

$\boxed{t \approx 11.6 \text{ days}}$

(b) What is the equivalent annual interest rate?

$P(1 + r_a)^t = Pe^{.06t}$ w/ $t = 1$

annual interest rate

$1 + r_a = e^{.06}$

$r_a = e^{.06} - 1 \approx .0618$ or $\boxed{6.18\%}$