Homework Solutions for 4/23/2010

5.4 #8 $(3x+2y)^{17}$ has terms $\binom{17}{j}(3x)^{17-j}(2y)^{j}$ and x^8y^9 is the case $j=9$. So the coefficient is $\boxed{\binom{17}{9}\cdot 3^8 \cdot 2^9}$ $\left(= \frac{17!\cdot 3^8 \cdot 2^9}{8!\,9!} = 81{,}662{,}929{,}920\right)$

5.4 #14 to make things easier, consider the cases n even, n odd separately.

n even: we want: $1 = \binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{n/2}$

and: $1 = \binom{n}{n} < \binom{n}{n-1} < \binom{n}{n-2} < \cdots < \binom{n}{n/2}$.

Note, second statement follows from first since $\binom{n}{j} = \binom{n}{n-j}$ for all j. We also know by direct computation that $\binom{n}{0} = 1$. It remains to show that $\binom{n}{j-1} < \binom{n}{j}$ for $j \le n/2$.

we have:

$$\frac{n!}{(j-1)!(n-j+1)!} < \frac{n!}{j!(n-j)!}$$

$$\iff$$

$$\frac{j!}{(j-1)!} < \frac{(n-j+1)!}{(n-j)!} \quad \text{(by cross-multiplying)}$$

$$\iff$$

$$j < n-j+1.$$

The last statement is true because $j \le n/2$. Hence $\binom{n}{j-1} < \binom{n}{j}$ for such j.

n odd: writing $n = 2k+1$, we want:

$$1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{k}$$
$$\qquad\qquad\qquad\qquad\qquad\quad \|$$
$$1 = \binom{n}{n} < \binom{n}{n-1} < \cdots < \binom{n}{k+1}$$

Again, the second line follows from the first, and proving the first line is exactly the same as before.

5.4 #20 Direct computation:

$$\text{Left side} = \frac{(n-1)!\; n!\; (n+1)!}{(k-1)!\,(n-k)!\,(k+1)!\,(n-k-1)!\,k!\,(n-k+1)!}$$

$$\text{Right side} = \frac{(n-1)!\; n!\; (n+1)!}{k!\,(n-k-1)!\,(k-1)!\,(n-k+1)!\,(k+1)!\,(n-k)!}$$

looking carefully, these are the same.

5.4 #24 Let $n = \binom{p}{k}$, which is a positive integer.

so $n = \dfrac{p!}{k!(p-k)!}$, hence $n \cdot k! \cdot (p-k)! = p!$.

Recall the theorem: if $p \mid abc$ and p is prime then $p \mid a$ or $p \mid b$ or $p \mid c$.

(Lemma 2, p. 233)

Since $p \mid p!$, we have $p \mid (n)(k!)((p-k)!)$.

By the theorem, it suffices to show that $p \nmid k!$ and $p \nmid (p-k)!$ to conclude $p \mid n$.

But $k!$ and $(p-k)!$ are products of numbers that are all less than p, so their prime factors are all less than p, hence p cannot divide $k!$ or $(p-k)!$

So $p \mid n$ (by the theorem quoted above).

5.4 #28 (a) Combinatorial version:

Let $S = A \cup B$, A, B disjoint sets each with n elements.

Then $\binom{2n}{2}$ = # ways of choosing a subset of S of size two.

Count these ways as follows. There are three separate (non-overlapping) cases:

- choose two from A
- choose two from B
- choose one from each.

The number of ways, in each case, is $\binom{n}{2}$, $\binom{n}{2}$, and $n \cdot n$ respectively. By the sum rule, the total number is $2\binom{n}{2} + n^2$.

Hence $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

(b) Algebra: $\binom{2n}{2} = \frac{(2n)!}{2!(2n-2)!} = \frac{2n(2n-1)}{2} = 2n^2 - n$

$2\binom{n}{2} + n^2 = 2\frac{n!}{2(n-2)!} + n^2 = n(n-1) + n^2 = 2n^2 - n$

So they are equal.

8.1 #4 (a) a is taller than b

- not reflexive (a is not taller than a)
- not symmetric
- antisymmetric (aRb and bRa is never true)
- transitive

(b) a and b were born on the same day

- reflexive
- symmetric
- not anti-symmetric
- transitive

(c) a has the same first name as b

- reflexive • symmetric • not anti-symmetric
- transitive

(d) a and b have a common grandparent

- reflexive • symmetric • not antisymmetric
- not transitive

8.1 #7 (a) $x \neq y$ • not reflexive • symmetric • not anti-symmetric • not transitive

(b) $xy \geq 1$ • not reflexive (take $x = y = 0$)

- symmetric

• not anti-symmetric

• transitive: since we are using integers, note that $xy \geq 1 \iff x,y \neq 0$ and x,y have the same sign.

Now it is easy to see that xRy and $yRz \Rightarrow xRz$.

(c) $x = y \pm 1$: • not reflexive • symmetric • not antisymmetric • not transitive

(d) $x \equiv y \pmod 7$ • reflexive • symmetric • not antisymmetric • transitive

(e) x is a multiple of y: • reflexive • not symmetric • not anti-symmetric (eg use $2,-2$) • transitive

(f) x,y both negative, or both nonnegative • reflexive • symmetric • not antisymmetric • transitive

(g) $x = y^2$ • not reflexive • not symmetric • antisymmetric • not transitive

↳ if $x = y^2$ and $y = x^2$ then $x = x^4$ so $x = 1$, so $y = 1$. Hence $x = y$.

(h) $x > y^2$ • not reflexive • not symmetric • antisymmetric • transitive

↳ "$x > y^2$ and $y > x^2$" is never true

8.4 #8 (a) The relation "$a=b$" is symmetric and antisymmetric.

(b) Let $A = \{a,b,c\}$ and let $R \subset A \times A$ be the relation $\{(a,b),(b,c),(c,b)\}$. Then R is not symmetric (since aRb and $b \not R a$) and is not antisymmetric (bRc and cRb and $b \neq c$).

8.4 #28

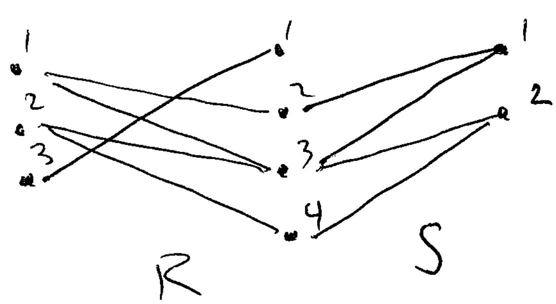
(a) $R_1 \cup R_2 = R_2$ since $R_1 \subset R_2$

(b) $R_1 \cap R_2 = R_1$ since $R_1 \subset R_2$

(c) $R_1 \oplus R_2 = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

(d) $R_2 - R_1$ = same.

8.4 #30



$S \circ R = \{(1,1),(1,2),(2,1),(2,2)\}$