Name: Id:

Final Exam
Math 2513-001
May 14, 2009

Problem 1:	Problem 4:
Problem 2:	Problem 5:
Problem 3:	Problem 6:

Total:

1(a) Determine whether 97 is prime, as efficiently as you can. State clearly any theorems that you use, and show your work.

(b) Determine the cardinality of each of these sets, with brief explanations:

(i) the set of all finite strings of 0s and 1s

(ii) the set of all functions from $\{1, 2, 3\}$ to $\{a, b, c, d\}$

(iii) the set of all irrational numbers

2(a) How many positive integers are there whose distinct prime factors are exactly 2, 3, 5, and 7, having 10 prime factors? (eg. $2 \cdot 3^4 \cdot 5^3 \cdot 7^2$)

2(b) Write the number 1023 in base 4. (Show your work.)

3. On a certain island, everyone is a knight or a knave. Knights always tell the truth and knaves always lie. A says "B is a knave." B says "the two of us are of opposite types." Let p be the proposition "A is a knight" and let q be the proposition "B is a knight."

(a) Write down two propositions (involving p and q) which express the information you have learned from the statements of A and B.

(b) Use truth tables for your propositions to determine what A and B are.

4(a) Show, by a combinatorial argument, that $\binom{2n}{2} = 2\binom{n}{2} + n^2$ for any $n \in \mathbf{Z}_+$.

(b) Show that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

5. Let $f\colon A \to B$ and $g\colon B \to C$ be functions.

(a) Define what it means for f to be *injective (one-to-one)*, *surjective (onto)*, and *bijective*.

(b) Prove that if $g \circ f$ is surjective then so is g.

(c) If $g \circ f$ is a bijection, is f a bijection? Explain why or why not.

6. Recall from the homework that if p is prime and $0 < k < p$ then p divides $\binom{p}{k}$.

(a) Give the precise definition of the statement "$\ell \equiv m \pmod{n}$."

(b) Using the fact above, prove that for any positive integers a and b and any prime p,

$$(a+b)^p \equiv a^p + b^p \pmod{p}.$$

Extra Page