**1.** Consider the two planes

$$x + y + z = 0, \qquad 3x + 2y + 2z = 0.$$

**(a)** Find the cosine of the angle between the two planes.
**(b)** Find a direction vector for the line of intersection between the two planes.
**(c)** Find a point that is on both planes.
**(d)** Find a vector equation for the line of intersection.

(a) normal vectors: $n_1 = \langle 1, 1, 1 \rangle$, $n_2 = \langle 3, 2, 2 \rangle$

angle between planes = angle between $n_1$ and $n_2$

$n_1 \cdot n_2 = |n_1||n_2| \cos\theta$

$n_1 \cdot n_2 = 3 + 2 + 2 = 7$, $|n_1| = \sqrt{3}$,
$|n_2| = \sqrt{9+4+4} = \sqrt{17}$.

so $\boxed{\cos\theta = \frac{7}{\sqrt{3}\sqrt{17}}}$

(b) It is perpendicular to both normal vectors

so use $\underline{v} = n_1 \times n_2$.

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 3 & 2 & 2 \end{vmatrix} = \langle 0, 3-2, 2-3 \rangle$$

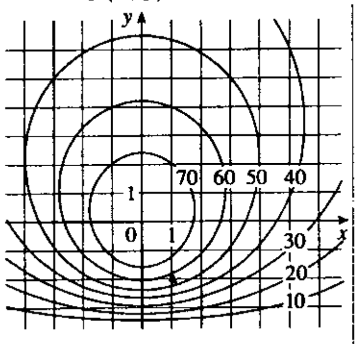
$\boxed{\underline{v} = \langle 0, 1, -1 \rangle}$

(c) The point $\boxed{(0, 0, 0)}$ satisfies both equations.

(d) $\boxed{\underline{r}(t) = \langle 0,0,0 \rangle + t\langle 0, 1, -1 \rangle}$

or $r(t) = \langle 0, t, -t \rangle$

**2(a)** The function $f(x, y)$ has level curves as shown in the picture below.




$\boxed{(i)\ f(1,-2) \approx 55}$
(between levels 50 and 60)

**(i)** estimate $f(1, -2)$, explain.
**(ii)** estimate $f_x(1, -2)$, explain. [think: rise/run]
**(iii)** estimate $f_y(1, -2)$, explain.

(ii) rise = change in $f$ as 1 move 1 unit in x-direction

run = 1 unit

so $f_x(1,-2) \approx \frac{-10}{1} = \boxed{-10}$

(iii) $f_y(1,-2) \approx \frac{10}{1/2} = \boxed{20}$ (increase of 10 as one moves $\frac{1}{2}$ in y-dir.)

**2(b)** Find all second partial derivatives of $f(x, y) = (x^2 + y)e^{-4y}$.

$f_x = 2xe^{-4y}$ $\qquad f_y = (x^2+y)(-4e^{-4y}) + e^{-4y}$

$\boxed{f_{xx} = 2e^{-4y}}$ $\qquad \boxed{f_{yy} = (x^2+y)(16e^{-4y}) + (-4e^{-4y}) + (-4e^{-4y})}$

$\boxed{f_{xy} = -8xe^{-4y}}$ $\qquad \boxed{f_{yx} = -8xe^{-4y}}$

**3.** Consider the vector-valued function

$$\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle, \quad -3 \le t \le 3.$$

**(a)** Find $\mathbf{r}'(t)$, $\mathbf{T}(t)$, $\mathbf{T}'(t)$, and the curvature $\kappa$ for the curve given by $\mathbf{r}(t)$.

**(b)** Find the length of the curve.

(a) $\boxed{r'(t) = \langle 1, -3\sin t, 3\cos t\rangle}$

$|r'(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{1+9} = \sqrt{10}$

$T(t) = \dfrac{r'(t)}{|r'(t)|} = \boxed{\dfrac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle}$

$\boxed{T'(t) = \dfrac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle}$

$|T'(t)| = \dfrac{1}{\sqrt{10}}\sqrt{0 + 9\cos^2 t + 9\sin^2 t}$

$= \dfrac{3}{\sqrt{10}}$

$\kappa = \dfrac{|T'(t)|}{|r'(t)|} = \dfrac{3/\sqrt{10}}{\sqrt{10}} = \boxed{\dfrac{3}{10}}$

(b)

$\text{Length} = \displaystyle\int_{-3}^{3} |r'(t)|\,dt = \int_{-3}^{3} \sqrt{10}\,dt$

$= \boxed{6\sqrt{10}}$

4. Consider the quadric surface

$$x^2 - y^2 + z^2 = 1.$$

(a) Draw some traces of the form $x = k$ in the $yz$ plane.

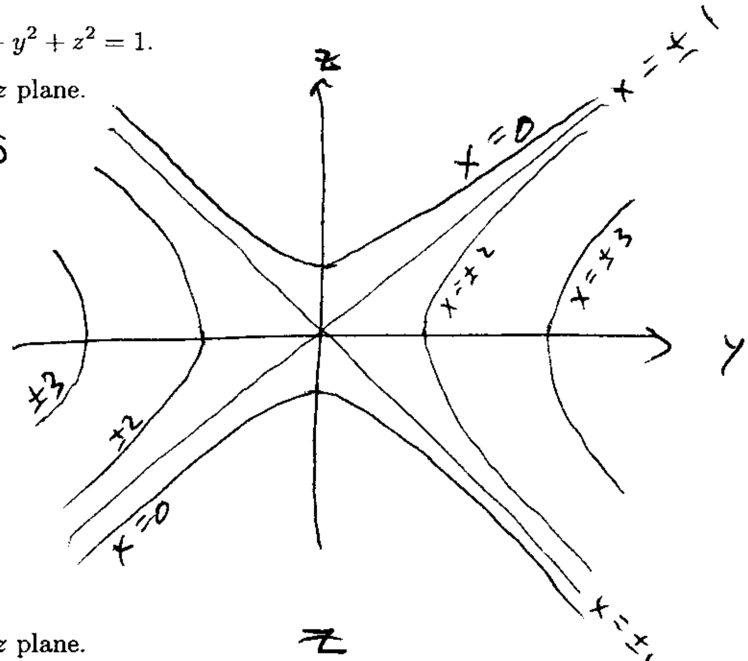
$z^2 - y^2 = 1 - k^2$, hyperbolas

$k = 0$: $z^2 - y^2 = 1$

$k = \pm 1$: $z^2 - y^2 = 0$

$k = \pm 2$: $z^2 - y^2 = -3$

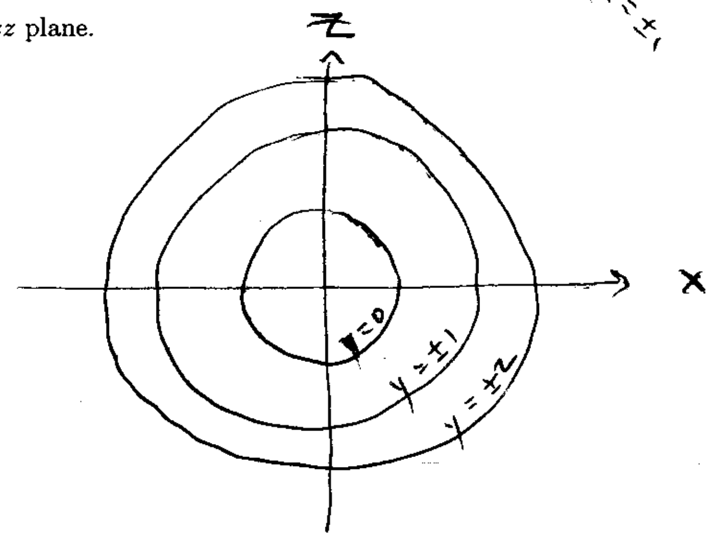
$k = \pm 3$: $z^2 - y^2 = -8$




(b) Draw some traces of the form $y = k$ in the $xz$ plane.

$x^2 + y^2 = 1 + k^2$

circles, radius $\geq 1$




(c) Draw some traces of the form $z = k$ in the $xy$ plane.

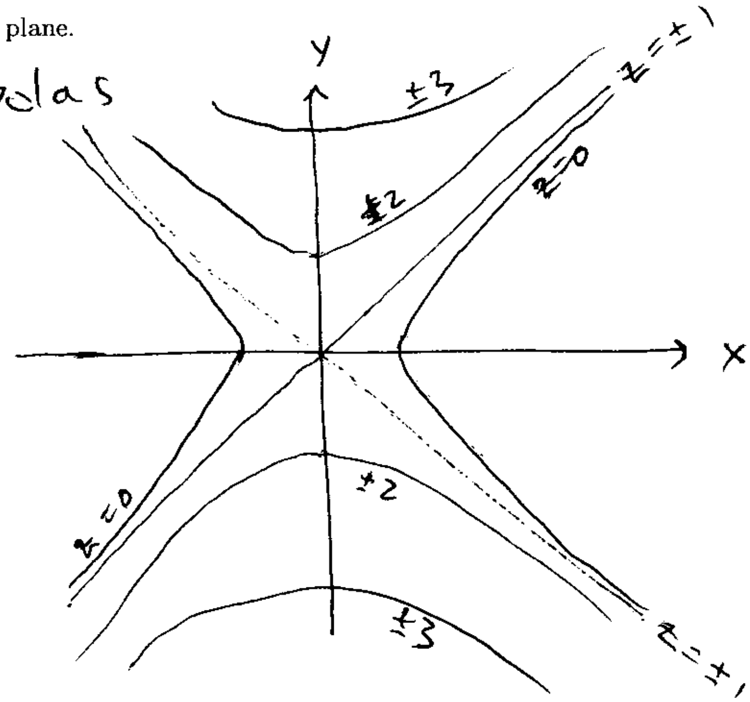
$x^2 - y^2 = 1 - k^2$, hyperbolas

$k = 0$: $x^2 - y^2 = 1$

$k = \pm 1$: $x^2 - y^2 = 0$

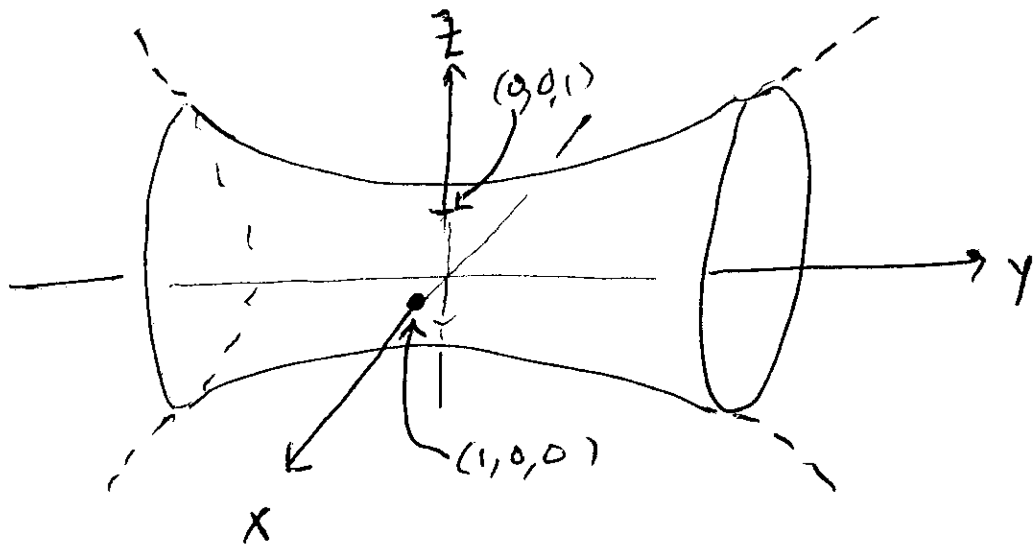
$k = \pm 2$: $x^2 - y^2 = -3$

$k = \pm 3$: $x^2 - y^2 = -8$



**(d)** What kind of surface is this? Do your best to draw a 3-d picture. Be sure to label your axes, indicate special points, etc.

hyperboloid of one sheet:




**Extra credit** Suppose $\mathbf{r}(t)$ has the property that $|\mathbf{r}(t)| = 4$ for all $t$. Give the argument that $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are always perpendicular.

We have $r(t) \cdot r(t) = |r(t)|^2 = 16$.

Take $\frac{d}{dt}$ of both sides:

$$\frac{d}{dt}(r(t) \cdot r(t)) = \frac{d}{dt}(16) = 0$$

$$\parallel$$

$$r(t) \cdot r'(t) + r'(t) \cdot r(t)$$

so $2\, r(t) \cdot r'(t) = 0$, so $r(t) \cdot r'(t) = 0$.

Since $v(t) = r'(t)$, we have that $r(t)$ and $v(t)$ are perpendicular.