1(a) Match the vector fields with their corresponding plots below.

I. $\mathbf{F}(x,y) = \langle 2, x \rangle$

II. $\mathbf{F}(x,y) = \langle 2x+2, y \rangle$

III. $\mathbf{F}(x,y) = \langle y, \cos x \rangle$

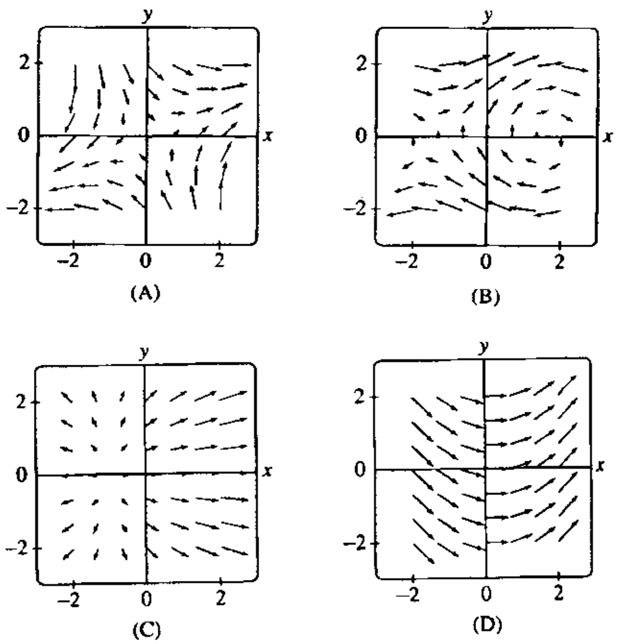
IV. $\mathbf{F}(x,y) = \langle x+y, x-y \rangle$

I = D

II = C

III = B

IV = A



1(b) Describe carefully in words the following surfaces (given with respect to spherical coordinates):

- $\rho = 5$ ← sphere of radius 5, centered at origin
- $\theta = \pi$ ← half-plane with edge along z-axis, and containing the negative x-axis
- $\phi = \pi/2$ ← the xy-plane

2(a) Let $\mathbf{F}(x,y,z) = \langle \frac{2xy}{z}, z + \frac{x^2}{z}, y - \frac{x^2 y}{z^2} \rangle$. Find a function $f(x,y,z)$ such that $\nabla f = \mathbf{F}$. Then, use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \langle t+1, t^2+2, t^3+3 \rangle$. $0 \le t \le 1$

$f_x = \frac{2xy}{z}$ says $f = \frac{x^2 y}{z} + C(y,z)$

$f_y = z + \frac{x^2}{z}$ says $f = zy + \frac{x^2 y}{z} + D(x,z)$

$f_z = y - \frac{x^2 y}{z^2}$ says $f = zy + \frac{x^2 y}{z} + E(x,y)$

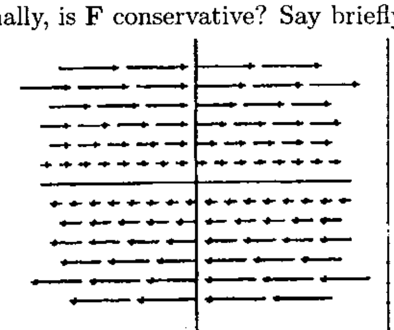
$\boxed{f(x,y,z) = zy + \frac{x^2 y}{z}}$ works (adding C is ok too)

$$\int_C \underline{F} \cdot d\underline{r} = f(\underline{r}(1)) - f(\underline{r}(0))$$
$$= f(2,3,4) - f(1,2,3)$$
$$= 12 + \frac{12}{4} - \left(6 + \frac{2}{3}\right)$$
$$= \boxed{\frac{25}{3}}$$

2(b) The figure shows a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$. For each of the following quantities, say whether it is positive, negative, or zero:

- $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(1,0)$ to $(1,2)$ zero
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(0,1)$ to $(2,1)$ positive
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle, oriented counter-clockwise negative
- $\frac{\partial P}{\partial x}$ zero
- $\frac{\partial P}{\partial y}$ positive
- $\frac{\partial Q}{\partial y}$ zero

Finally, is $\mathbf{F}$ conservative? Say briefly why or why not.

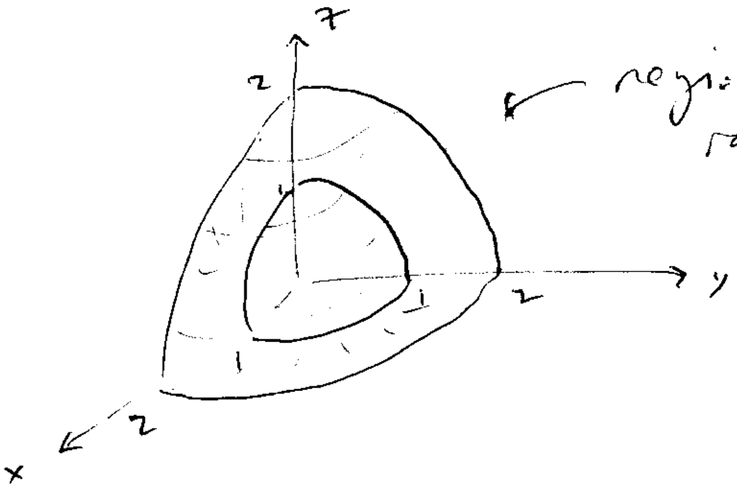


Not conservative, b/c integrating along the unit circle does not give zero.

3. Let E be the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant. (Recall, the *first octant* is the region where $x \geq 0$, $y \geq 0$, and $z \geq 0$.)

(a) Draw the region and describe it using spherical coordinates.

(b) Evaluate the integral $\iiint_E z\,dV$.



region between spheres of radius 1, radius 2

($\frac{1}{8}$ of a melon)

$$\boxed{\begin{aligned} 1 &\leq \rho \leq 2 \\ 0 &\leq \theta \leq \pi/2 \\ 0 &\leq \varphi \leq \pi/2 \end{aligned}}$$

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \underbrace{\rho\cos\varphi}_{z}\ \underbrace{\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi}_{dV}$$

$$= \int_1^2 \rho^3\,d\rho \int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi \int_0^{\pi/2} d\theta$$

$$= \left[\tfrac{1}{4}\rho^4\Big|_1^2 \right]\left[\int_0^1 u\,du \right]\left[\pi/2 \right]$$

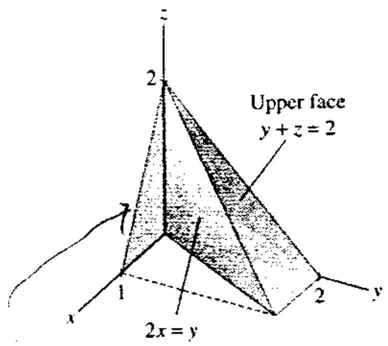
$u = \sin\varphi$
$du = \cos\varphi\, d\varphi$

$$= \left(4 - \tfrac{1}{4}\right)\left(\tfrac{1}{2}u^2\Big|_0^1\right)\left(\pi/2\right)$$

$$= \frac{15}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \boxed{\frac{15\pi}{16}}$$

4. The figure below shows the region E bounded by $y+z=2$, $2x=y$, $x=0$, and $z=0$. Express $\iiint_E xe^z\, dV$ in three ways, using $dV = dz\, dx\, dy$, $dV = dy\, dz\, dx$, and $dV = dx\, dy\, dz$. Do *not* solve the integrals.



line: $z = 2 - 2x$

① $\displaystyle\int_0^2 \int_0^{y/2} \int_0^{2-y} xe^z \, dz\, dx\, dy$

② $\displaystyle\int_0^1 \int_0^{2-2x} \int_{2x}^{2-z} xe^z \, dy\, dz\, dx$

③ $\displaystyle\int_0^2 \int_0^{2-z} \int_0^{y/2} xe^z \, dx\, dy\, dz$

5. Evaluate the following integrals:

(a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle y^2, xy \rangle$ and C is given by $\mathbf{r}(t) = \langle t^3, t^4 \rangle$, $0 \le t \le 4$.

$$r'(t) = \langle 3t^2, 4t^3 \rangle$$

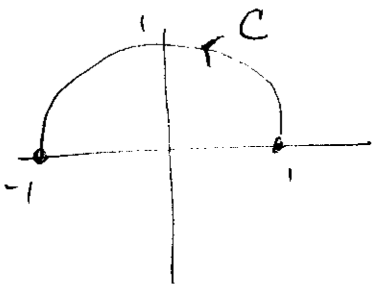
$$\int_0^4 \langle (t^4)^2, (t^3)(t^4) \rangle \cdot \langle 3t^2, 4t^3 \rangle \, dt$$

$$= \int_0^4 (3t^{10} + 4t^{10}) \, dt$$

$$= \int_0^4 7t^{10} \, dt = \frac{7}{11} t^{11} \Big|_0^4$$

$$= \boxed{\frac{7}{11}(4)^{11}}$$

(b) $\int_C xy^2 \, ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$, traversed in the counter-clockwise direction.



use $r(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$

then $r'(t) = \langle -\sin t, \cos t \rangle$

$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

so $ds = |r'(t)| dt = dt$

$$\int_C xy \, ds = \int_0^\pi \cos t \sin^2 t \, dt \qquad u = \sin t, \quad du = \cos t \, dt$$

$$= \int_0^0 u^2 \, du = \boxed{0}$$