Take-home Exam I
Algebraic Topology
Due October 19, 2006

1. Let X be path connected, locally path connected, and semilocally simply connected. Let H_0 and H_1 be subgroups of $\pi_1(X, x_0)$ such that $H_0 \subset H_1$. Let $p_i \colon X_{H_i} \to X$ be covering spaces corresponding to the subgroups H_i. Prove that there is a covering space map $f \colon X_{H_0} \to X_{H_1}$ such that $p_1 \circ f = p_0$.

---

2. Show that if a path connected, locally path connected space X has finite fundamental group, then every map $X \to S^1$ is nullhomotopic. [Use the covering $\mathbb{R} \to S^1$.]

---

3. Let a and b be the two free generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands.
(a) Find the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by $\{a^2, b^2\}$.
(b) Find the covering space corresponding to the normal subgroup generated by $\{a^2, b^2, (ab)^4\}$.

---

4. Find all connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism without basepoints.

---

5. Let $p \colon \widetilde{X} \to X$ be a simply connected covering space of X. Let $A \subset X$ be path connected and locally path connected, and let $\widetilde{A} \subset \widetilde{X}$ be a path component of $p^{-1}(A)$. Show that the restricted map $p \colon \widetilde{A} \to A$ is the covering space corresponding to the kernel of the homomorphism $\pi_1(A) \to \pi_1(X)$.

---

6. Let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\phi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 - \{0\}$. Show this action is a covering space action. Show the orbit space $X/\mathbb{Z}$ is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis. Can you find the fundamental group of $X/\mathbb{Z}$?