Exam I Solutions
Algebraic Topology
October 19, 2006

1. Let X be path connected, locally path connected, and semilocally simply connected. Let H_0 and H_1 be subgroups of $\pi_1(X, x_0)$ such that $H_0 \subset H_1$. Let $p_i\colon X_{H_i} \to X$ be covering spaces corresponding to the subgroups H_i. Prove that there is a covering space map $f\colon X_{H_0} \to X_{H_1}$ such that $p_1 \circ f = p_0$.

Let $y_i \in X_{H_i}$ be basepoints above x_0 ($i = 0, 1$). Then $p_{i*}(\pi_1(X_{H_i}, y_i)) = H_i$, and $H_0 \subset H_1$ implies there is a lift $f\colon X_{H_0} \to X_{H_1}$ of the map p_0, by the lifting criterion. That is, we have $f\colon X_{H_0} \to X_{H_1}$ such that $p_1 \circ f = p_0$.

We need to show that X_{H_1} is evenly covered by f. Given $y \in X_{H_1}$ let $U \subset X$ be a path connected open neighborhood of $x = p_1(y)$ that is evenly covered by both p_0 and p_1. Let $V \subset X_{H_1}$ be the slice of $p_1^{-1}(U)$ containing y. Note that V is path connected, and so are all slices of $p_1^{-1}(U)$ and $p_0^{-1}(U)$.

Denote the slices of $p_0^{-1}(U)$ by $\{W_z \mid z \in p_0^{-1}(x)\}$, where W_z is the slice containing the point z. Let C be the subcollection $\{W_z \mid z \in f^{-1}(y)\}$. Every slice W_z is mapped by f into a single slice of $p_0^{-1}(U)$, since these sets are path connected. Since $f(z) \in p_1^{-1}(x)$, the image of W_z is in V if and only if $f(z) = y$. Hence $f^{-1}(V)$ is the union of the slices in C.

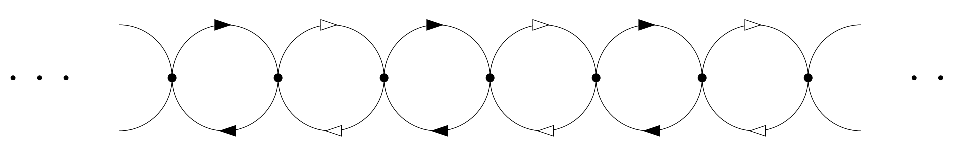
Given $W_z \in C$, the homeomorphisms $p_0|_{W_z}$ and $p_1|_V \circ f|_{W_z}$ are equal, and so $(p_1|_V)^{-1} \circ p_0|_{W_z} = f|_{W_z}$. Hence the latter map is a homeomorphism, and V is an evenly covered neighborhood of y.

2. Show that if a path connected, locally path connected space X has finite fundamental group, then every map $X \to S^1$ is nullhomotopic. [Use the covering $\mathbb{R} \to S^1$.]

Let $f\colon X \to S^1$ be the map and $p\colon \mathbb{R} \to S^1$ the usual covering map. The image subgroup $f_*(\pi_1(X)) \subset \pi_1(S^1)$ is finite, and must therefore be trivial since $\mathbb{Z}$ has no non-trivial finite subgroups. Then $f_*(\pi_1(X)) \subset p_*(\pi_1(\mathbb{R}))$ (for any choice of basepoints), so the lifting criterion implies that there is a lift $\tilde{f}\colon X \to \mathbb{R}$. Let $F\colon X \times I \to \mathbb{R}$ be the straight line homotopy from $\tilde{f}$ to any constant map. Then $p \circ F$ is a homotopy from f to a constant map, and f is nullhomotopic.

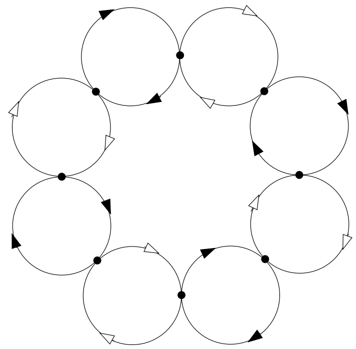
3. Let a and b be the two free generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands.
(a) Find the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by $\{a^2, b^2\}$.
(b) Find the covering space corresponding to the normal subgroup generated by $\{a^2, b^2, (ab)^4\}$.

(A) *The covering is shown below:*



Each vertex is in the same orbit as its neighbor, via a covering translation given by rotation by π in the circle through the two vertices. Thus, the covering group acts transitively on a fiber, and the covering is regular. Since a^2 and b^2 are loops in the cover, the corresponding normal subgroup *of* $\langle a, b\rangle$ *contains* $\langle\langle a^2, b^2\rangle\rangle$ *(the smallest normal subgroup containing* a^2 *and* b^2*). For the opposite inclusion, let* T *be the maximal tree given by all the leftward-oriented edges in the picture. Using this tree in the usual way, the free generators of the fundamental group are represented by the loops with labels* $(ab)^k a^2 (ab)^{-k}$ *or* $(ab)^k ab^2 a^{-1}(ab)^{-k}$ *for* $k \in \mathbb{Z}$*. These words are all conjugates in* $\langle a, b\rangle$ *of* a^2 *and* b^2*, so the image subgroup is contained in* $\langle\langle a^2, b^2\rangle\rangle$*.*

(B) *The covering is:*



As above, the covering is regular since the group of covering translations acts transitively on the vertices. (One of these is given by the composition of a reflection in the plane, and "inversion" which switches inside and outside edges.) Since a^2*,* b^2*, and* $(ab)^4$ *are all represented by loops in the cover, the corresponding normal subgroup of* $\langle a, b\rangle$ *contains* $\langle\langle a^2, b^2, (ab)^4\rangle\rangle$*. Next let* T *be the maximal tree consisting of all inner edges except for one "b" edge. The fundamental group of the cover has 9 free generators, labeled by the words* a^2*,* ab^2a^{-1}*,* $aba^2(ab)^{-1}$*,* $abab^2(aba)^{-1}$*,* $ababa^2(abab)^{-1}$*,* $ababab^2(ababa)^{-1}$*,* $abababa^2(ababab)^{-1}$*,* $abababab^2(abababa)^{-1}$*, and* $(ab)^4$*. These are all conjugates of* a^2*,* b^2*, and* $(ab)^4$*, so the subgroup is contained in* $\langle\langle a^2, b^2, (ab)^4\rangle\rangle$*.*

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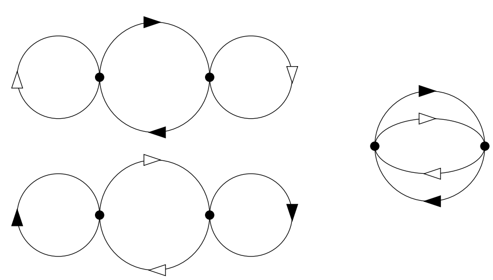
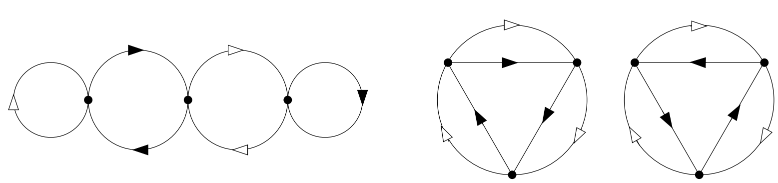
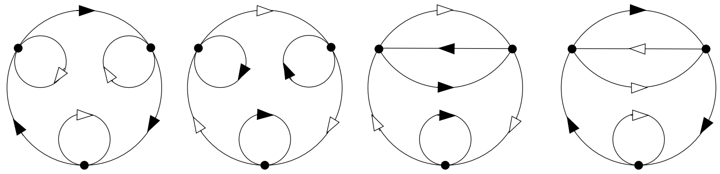
4. Find all connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism without basepoints.

There are three 2-sheeted coverings:



There are seven 3-sheeted coverings:





5. Let $p\colon \widetilde{X} \to X$ be a simply connected covering space of X. Let $A \subset X$ be path connected and locally path connected, and let $\widetilde{A} \subset \widetilde{X}$ be a path component of $p^{-1}(A)$. Show that the restricted map $p\colon \widetilde{A} \to A$ is the covering space corresponding to the kernel of the homomorphism $\pi_1(A) \to \pi_1(X)$.

We have seen in the previous course that $p|_{p^{-1}(A)}\colon p^{-1}(A) \to A$ is a covering space. Also, it is easy to check that the restriction of a covering space to a path component is also a covering space. Now let $x_0 \in A$ and $y_0 \in \widetilde{A}$ be basepoints with $p(y_0) = x_0$.

Let $i\colon A \to X$ and $j\colon \widetilde{A} \to \widetilde{X}$ be the inclusion maps and let $p' = p|_{\widetilde{A}}$. Then we have $p \circ j = i \circ p'$, and hence $p_* \circ j_* = i_* \circ p'_*$. Since $p_* \circ j_*$ factors through $\pi_1(\widetilde{X}, y_0)$, which is trivial, the composition $i_* \circ p'_*$ is trivial. That is, $p'_*(\pi_1(\widetilde{A}, y_0)) \subset \ker(i_*)$.

Next we show that $p'_*(\pi_1(\widetilde{A}, y_0)) \supset \ker(i_*)$. Let $[f] \in \ker(i_*)$ where f is a loop in A at x_0. Let $\tilde{f}$ be the unique lift of f to $\widetilde{A}$ with initial endpoint y_0. Then $j \circ \tilde{f}$ is a lift of $i \circ f$ to $\widetilde{X}$ with initial point y_0. Since $[i \circ f] = 1$ in $\pi_1(X, x_0)$, this latter lift is in fact a loop in $\widetilde{X}$. Hence $\tilde{f}$ is a loop in $\widetilde{A}$, and $[f] = [p' \circ \tilde{f}]$ in $\pi_1(A, x_0)$. That is, $[f]$ is the image of $[\tilde{f}]$ under p'_*.

6. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\phi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 - \{0\}$. Show this action is a covering space action. Show the orbit space $X/\mathbb{Z}$ is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis. Can you find the fundamental group of $X/\mathbb{Z}$?

First we need to show that every point has on open neighborhood that is disjoint from all of its translates under powers of ϕ. For the point (a, b) a product neighborhood of the form $(c, 2c) \times \mathbb{R}$ where $a/2 < c < a$ (if $a \neq 0$) or $\mathbb{R} \times (d, 2d)$ where $b/2 < d < b$ (if $b \neq 0$) will work. Hence $q\colon X \to X/\mathbb{Z}$ is a covering space map.

To see that $X/\mathbb{Z}$ is not Hausdorff, consider two points of the form $(a, 0)$ and $(0, b)$ in X. Any open neighborhood of $[(a, 0)]$ in $X/\mathbb{Z}$ is given by a ϕ-invariant open neighborhood of the orbit of $(a, 0)$. Such a set must contain a small product neighborhood $U = (a - \epsilon, a + \epsilon) \times (-\epsilon, \epsilon)$ and all of its translates under powers of ϕ. Simlarly a neighborhood of $[(0, b)]$ is given by a ϕ-invariant set containing $V = (-\delta, \delta) \times (b - \delta, b + \delta)$ and its images under ϕ^k, $k \in \mathbb{Z}$. Now take $k \in \mathbb{Z}$ large enough that $a/2^k < \delta$ and $2^k\epsilon > b$. Then $\phi^{-k}(U)$ intersects V, and so $[(a, 0)]$ and $[(0, b)]$ cannot be separated by open sets in $X/\mathbb{Z}$.

The group action preserves the subset $(0, \infty) \times \mathbb{R}$, and identifies each line $x = a$ homeomorphically to the line $x = 2a$. Hence the image of this set is homeomorphic to $S^1 \times \mathbb{R}$. The same is true of the subsets $(-\infty, 0) \times \mathbb{R}$, $\mathbb{R} \times (0, \infty)$, and $\mathbb{R} \times (-\infty, 0)$. Since these sets cover X, the quotient is a union of four copies of $S^1 \times \mathbb{R}$. Each line of the form $\{a\} \times \mathbb{R}$ in the annulus $((0, \infty) \times \mathbb{R})/\mathbb{Z}$ spirals around and limits onto two circles (the images of the two halves of the y-axis). Similarly, the circle $S^1 \times \{0\}$ is the limiting circle for lines in two of the other annuli.

We know that S has fundamental group $\mathbb{Z}$ and the group of covering translations of this (regular) covering is also $\mathbb{Z}$. Hence $\pi_1(X/\mathbb{Z})$ maps onto $\mathbb{Z}$ with kernel isomorphic to $\mathbb{Z}$. It follows that $\pi_1(X/\mathbb{Z})$ is a semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$. There are only two such groups, $\mathbb{Z} \times \mathbb{Z}$ and a non-abelian group (because there are only two automorphisms of $\mathbb{Z}$). To see that $\pi_1(X/\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$, we show that its generators commute. These generators are given by loops in $X/\mathbb{Z}$ as follows: one is the image of the loop γ in X which generates $\pi_1(X, x_0)$; the other is the image of a path α in X joining the basepoint x_0 to $\phi(x_0)$. It is not difficult to map $I \times I$ into X so that its boundary maps to the path $\gamma\alpha\phi(\overline{\gamma})\overline{\alpha}$. This is possible because ϕ preserves the orientation of X. Then the boundary of the image of this square in $X/\mathbb{Z}$ represents the commutator of the generators, and so they commute.