Final Exam
Algebraic Topology
December 13, 2006

You may apply theorems from the course, but please give the name or statement of the theorem.

1. Compute the homology groups of $S^3 \times S^5$ by using a product cell structure and cellular homology.

There are cell structures for S^3 and S^5 having cells e^0, e^3 and e^0, e^5 respectively. Hence the product cell structure for $S^3 \times S^5$ has cells $e^0 \times e^0$, $e^0 \times e^5$, $e^3 \times e^0$, and $e^3 \times e^5$. That is, there is one cell of each of the dimensions $0, 3, 5, 8$. Thus the cellular chain complex for $S^3 \times S^5$ is

$$\cdots \to 0 \to \mathbb{Z} \to 0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to 0 \to \mathbb{Z} \to 0.$$

Note that all homomorphisms are zero, so the homology groups of this chain complex are equal to the chain groups. So $H_i(S^3 \times S^5)$ is $\mathbb{Z}$ for $i = 0, 3, 5, 8$ and 0 otherwise.

2. Find the local homology groups $H_n(X, X - \{x\})$ for various points x in the Möbius band and in the annulus $S^1 \times I$. Then show that these two spaces are not homeomorphic. [Consider their boundaries.]

By Excision, $H_n(X, X - \{x\}) \cong H_n(U, U - \{x\})$ for any open neighborhood U of $x \in X$. Also recall that $H_n(U, U - \{x\}) = \widetilde{H}_n(U, U - \{x\})$. To compute $\widetilde{H}_n(U, U - \{x\})$ we consider two possibilities for x.

If x is an interior point of the Möbius band or annulus, U may be taken to be homeomorphic to an open disk with center x. Since the disk is contractible, the long exact sequence for the pair $(U, U - \{x\})$ gives that the boundary map $\partial\colon \widetilde{H}_n(U, U - \{x\}) \to \widetilde{H}_{n-1}(U - \{x\})$ is an isomorphism for all n. Note that $U - \{x\}$ is homotopy equivalent to a circle, with reduced homology groups given by $\mathbb{Z}$ in dimension 1 and zero otherwise. Thus $\widetilde{H}_n(U, U - \{x\})$ is $\mathbb{Z}$ for $n = 2$ and zero for $n \neq 2$.

If x is a boundary point of the Möbius band or annulus, U may be taken to be homeomorphic to half of an open disk, such as $\{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 < 1 \text{ and } b \geq 0\}$, with x corresponding to the origin. In this case both U and $U - \{x\}$ are contractible, so every two out of three groups in the reduced long exact sequence for $(U, U - \{x\})$ is zero. Exactness implies that $\widetilde{H}_n(U, U - \{x\}) = 0$ for all n.

Thus boundary points and interior points have different local homology groups in dimension 2. Hence any homeomorphism must restrict to homeomorphisms of the boundaries and of the interiors. Now, to see that there is no homeomorphism from the Möbius band to the annulus, note that their boundaries are not homeomorphic, since the boundary of the Möbius band is a circle and the boundary of the annulus is two circles.

3. Recall that the *augmentation map* $\varepsilon\colon C_0(X) \to \mathbb{Z}$ take a 0-chain $\sum_i n_i \sigma_i$ to the integer $\sum_i n_i$. Prove that if X is non-empty and path connected then ε induces an isomorphism $H_0(X) \to \mathbb{Z}$.

Since X is non-empty, there exists a singular 0-simplex $\sigma\colon \Delta^0 \to X$. Then $\varepsilon(n\sigma) = n$, and ε is surjective. So $C_0(X)/\ker(\varepsilon) \xrightarrow{\varepsilon} \mathbb{Z}$ is an isomorphism, and it remains to verify that $\ker(\varepsilon)$ is equal to the image of $\partial_1\colon C_1(X) \to C_0(X)$.

If $\sigma\colon \Delta^1 \to X$ is a 1-simplex (a path), then $\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma|_{v_1} - \sigma|_{v_0}) = 1 - 1 = 0$. Hence $\mathrm{im}(\partial_1) \subset \ker(\varepsilon)$, since 1-simplices generate $C_1(X)$.

If $\alpha = \sum_i n_i\sigma_i$ is a 0-chain with $\sum_i n_i = 0$ then we construct a 1-chain with boundary α as follows. Let $x_i \in X$ be the image of the 0-simplex σ_i. If n_i is negative, draw $|n_i|$ outgoing paths from x_i, and if n_i is positive, draw n_i incoming paths to x_i. Since $\sum_i n_i = 0$, the numbers of incoming and outgoing paths agree, and they can be joined in a bijective fashion (since X is path connected). That is, we now have a family of paths, such that the number of endpoints at x_i is $|n_i|$, with signs corresponding to orientations. This family of paths, considered as a 1-chain, has boundary α. Hence $\mathrm{im}(\partial_1) \supset \ker(\varepsilon)$. [See Proposition 2.7 of Hatcher for a slightly different argument.]

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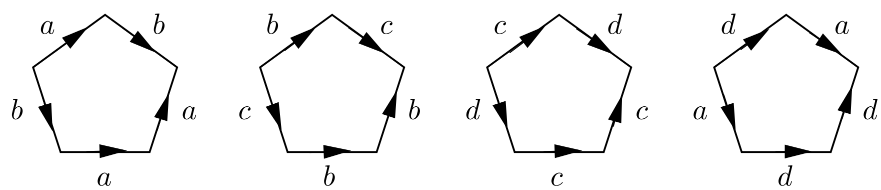
4. (a) Using cellular homology, find the homology groups of the 2-complex X which is defined as follows. It has one 0-cell, four 1-cells (a, b, c, and d), and four 2-cells attached to the 1-skeleton as shown below:



(b) Write down a presentation for the fundamental group of X. [It turns out that this group is infinite, though this is not at all obvious.]

(A) *First note that the cellular chain complex has the form $0 \to \mathbb{Z}^4 \xrightarrow{\partial_2} \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z} \to 0$, by considering the number of cells of each dimension. Since each 1-cell has both endpoints mapping to the same 0-cell (with opposite signs), the homomorphism ∂_1 is zero. To compute ∂_2, we consider the attaching maps of each 2-cell and record its degree relative to each 1-cell. For the first 2-cell, its degrees around relative to a, b, c, and d, respectively, are $(1, 0, 0, 0)$. The second 2-cell has degrees $(0, 1, 0, 0)$, the third has degrees $(0, 0, 1, 0)$, and the fourth has degrees $(0, 0, 0, 1)$. Hence ∂_2 is given by the identity matrix.*

Now we can compute the homology groups of X. The chain complex is exact in dimensions 1 and higher, so $H_i(X) = 0$ for $i > 0$. Also, $H_0(X) = \mathbb{Z}$. Note that X has the same homology as a point.

(B) *A presentation is given by:*

$$\langle a, b, c, d \mid a^2b^{-1}a^{-1}b = b^2c^{-1}b^{-1}c = c^2d^{-1}c^{-1}d = d^2a^{-1}d^{-1}a = 1\rangle.$$

It turns out that this group is infinite (and non-trivial) and so X is not contractible, even though it has the homology of a contractible space.

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5. (a) Given $n > 1$, construct a space X such that $H_1(X)$ is cyclic of order n.

(b) Can you construct a space Y such that $H_2(Y)$ is cyclic of order n?

(A) *Let X be a cell complex having one 0-cell, one 1-cell, and one 2-cell. Attach the 2-cell by a map $S^1 \to S^1$ of degree n. The cellular chain complex is $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$. The first homology group is then $\mathbb{Z}/n\mathbb{Z}$.*

(B) *Let Y have one cell in each of dimensions 0, 2, and 3. Note that the 2-skeleton is homeomorphic to S^2. Attach the 3-cell by a map $S^2 \to S^2$ of degree n. Then the cellular chain complex is $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$ and the second homology group is $\mathbb{Z}/n\mathbb{Z}$.*

Alternatively, let $Y = SX$, the suspension of X. Then we have seen that $\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(Y)$, and so $H_2(Y) = \widetilde{H}_2(Y) \cong \widetilde{H}_1(X) \cong \mathbb{Z}/n\mathbb{Z}$.

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6. (a) Define the *degree* of a map $f\colon S^n \to S^n$.

(b) Prove that the antipodal map is not homotopic to the identity map if n is even. State carefully each of the properties of the degree that you use.

(A) *Given $f\colon S^n \to S^n$, there is an induced homomorphism $f_*\colon H_n(S^n) \to H_n(S^n)$. Since $H_n(S^n)$ is infinite cyclic, choosing a generator identifies $H_n(S^n)$ with $\mathbb{Z}$. Then $f_*\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by some integer, which is the degree of f.*

(B) *We regard S^n as the unit sphere in $\mathbb{R}^{n+1}$ with coordinates $x_0, \ldots, x_n$. The antipodal map sends $(x_0, \ldots, x_n)$ to $(-x_0, \ldots, -x_n)$. It is the composition of $n+1$ reflections across hyperplanes, each of which changes the sign of one coordinate.*

The properties we need are: $\deg(\mathrm{id}) = 1$, $\deg(r) = -1$ if r is reflection across a hyperplane, $\deg(f \circ g) = \deg(f)\deg(g)$, and $\deg(f) = \deg(g)$ if $f \simeq g$.

By the description of the antipodal map given above, its degree is $(-1)^{n+1}$, which is -1 if n is even. Then it cannot be homotopic to the identity map, by the first and last properties just given.
