Take-home Exam I
Algebraic Topology
Due March 6, 2007

1. Compute the simplicial cohomology groups of the Klein bottle and $\mathbb{R}P^2$ in $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients, using the Δ-complex structure given on page 102.

2. Regarding a cochain $\varphi \in C^1(X;G)$ as a function from paths in X to G, show that if φ is a cocycle, then
(a) $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$,
(b) φ takes the value 0 on constant paths,
(c) $\varphi(f) = \varphi(g)$ if $f \simeq g$,
(d) φ is a coboundary if and only if $\varphi(f)$ depends only on the endpoints of f, for all f.

3. Compute $H^i(S^n;G)$ in two ways:
(a) Show that $\widetilde{H}^{i-1}(S^{n-1};G) \cong \widetilde{H}^i(S^n;G)$ for all i, using either the long exact sequence or the Mayer-Vietoris sequence; then compute $H^i(S^n;G)$ by induction.
(b) Compute $H^i(S^n;G)$ using cellular cohomology.

4. Show that if $f\colon S^n \to S^n$ is a map of degree d then $f^*\colon H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d.

5. Show that if $\alpha\colon H \to H$ is multiplication by n then $\alpha^*\colon \operatorname{Ext}(H,G) \to \operatorname{Ext}(H,G)$ is also multiplication by n. [Hint: You'll need a chain map from a free resolution of H to itself. Instead of appealing to Lemma 3.1, try to guess what the chain map could be, and verify that it is in fact a chain map.]

6. Without using the Universal Coefficient theorem directly, prove that if (X,A) is a good pair then the quotient map $q\colon (X,A) \to (X/A, A/A)$ induces isomorphisms $q^*\colon H^n(X/A, A/A;G) \to H^n(X,A;G)$.