Take-home Exam II
Algebraic Topology
Due April 26, 2007

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1. Let I be a directed set and let $\{A_\alpha, f_{\alpha\beta}\}$ and $\{B_\alpha, g_{\alpha\beta}\}$ be directed systems of groups indexed over I. Suppose there are homomorphisms $\phi_\alpha\colon A_\alpha \to B_\alpha$ for each $\alpha \in I$ such that $g_{\alpha\beta}\circ\phi_\alpha = \phi_\beta\circ f_{\alpha\beta}$ for all α, β. That is, the maps ϕ_α form a "directed chain map" from $\{A_\alpha\}$ to $\{B_\alpha\}$.

Show that there is an induced homomorphism $\phi\colon \varinjlim A_\alpha \to \varinjlim B_\alpha$ making the following diagrams commute for all α:

$$\begin{array}{ccc} A_\alpha & \longrightarrow & \varinjlim A_\alpha \\ \downarrow{\scriptstyle \phi_\alpha} & & \downarrow{\scriptstyle \phi} \\ B_\alpha & \longrightarrow & \varinjlim B_\alpha \end{array}$$

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2. For each α let $\mathscr{C}_\alpha$ be a chain complex with groups $\{C^i_\alpha\}$ and boundary maps $\partial^i_\alpha\colon C^i_\alpha \to C^{i-1}_\alpha$. Suppose, for each i, that the groups $\{C^i_\alpha\}_{\alpha\in I}$ form a directed system together with maps $f^i_{\alpha\beta}\colon C^i_\alpha \to C^i_\beta$ for all $\alpha \leq \beta$. Suppose also that the boundary maps $\{\partial^i_\alpha\}_{\alpha\in I}$ form a "directed chain map" from $\{C^i_\alpha\}$ to $\{C^{i-1}_\alpha\}$ (see problem 1). Let $\partial^i\colon \varinjlim C^i_\alpha \to \varinjlim C^{i-1}_\alpha$ be the induced map given by problem 1.

(a) Show that these limit groups and maps form a chain complex. (Hence there is a limit chain complex $\varinjlim \mathscr{C}_\alpha$.)

(b) Show that if each $\mathscr{C}_\alpha$ is exact then so is $\varinjlim \mathscr{C}_\alpha$.

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3. For a map $f\colon M \to N$ between connected closed orientable manifolds with fundamental classes $[M]$ and $[N]$, the *degree* of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

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4. Show that $H^0_c(X; G) = 0$ if X is path-connected and noncompact.

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5. If M is a connected compact orientable n-manifold, a homeomorphism $f\colon M \to M$ is *orientation preserving* if f_* takes the fundamental class to itself, and *orientation reversing* otherwise. Recall that $\mathbb{C}\mathrm{P}^2$ is a 4-manifold with a cell structure having one cell each in dimensions $0, 2, 4$. We know that $H_4(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$, so it's orientable. Use the cup product to show that every homeomorphism $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ is orientation preserving. [Hint: Consider what f^* might do to a generator of $H^2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$. You may also want to use problem 4 from Exam I.]

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6. Show that if a connected closed orientable manifold M of dimension $2k$ has $H_{k-1}(M; \mathbb{Z})$ torsion-free, then $H_k(M; \mathbb{Z})$ is also torsion-free.