Final Exam
Algebraic Topology
May 10, 2007

You may apply theorems from the course, but please give the name or statement of the theorem.

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1. Consider the long exact sequence of homotopy groups

$$\cdots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \cdots$$

for a triple (X, A, B). Show that the sequence is exact at the $\pi_n(X, B, x_0)$ term.

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2. **(a)** State the Lefschetz fixed point theorem, and define the Lefschetz number $\tau(f)$ of a map $f\colon X \to X$, where X is a (retract of a) finite simplicial complex. (Note that this includes compact CW complexes.)

(b) Give the homology groups and cohomology groups of $\mathbb{C}\mathrm{P}^n$ in $\mathbb{Z}$-coefficients, and also describe the cup product structure (no proof needed).

(c) If $f^*\colon H^i(\mathbb{C}\mathrm{P}^n) \to H^i(\mathbb{C}\mathrm{P}^n)$ is multiplication by d, what is the map $f_*\colon H_i(\mathbb{C}\mathrm{P}^n) \to H_i(\mathbb{C}\mathrm{P}^n)$? Explain why, using the universal coefficient theorem.

(d) Prove that every map $f\colon \mathbb{C}\mathrm{P}^{2k} \to \mathbb{C}\mathrm{P}^{2k}$ has a fixed point. [Hint: use the cup product.]

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3. Prove the *extension lemma*: Let (X, A) be a finite-dimensional CW pair and Y a path connected space such that $\pi_{n-1}(Y) = 0$ for all n for which $X - A$ has an n-cell. Then every map $f\colon A \to Y$ can be extended to a map $X \to Y$. [Hint: define the extension cell-by-cell, in increasing dimensions.]

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4. Let (X, A) be a CW pair with A contractible. Use the extension lemma to show that A is a retract of X.

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5. Let $K \subset S^3$ be a *knot*, i.e. an embedded circle. Let N be a closed ε-neighborhood of K which is homeomorphic to the solid torus $D^2 \times S^1$. Let $X = S^3 - \mathrm{int}(N)$. Note that N and X are both compact 3-manifolds with boundary, with common boundary $X \cap N$, which is a torus.

Use the Mayer-Vietoris sequence to compute the first homology group of X. [It turns out the answer does not depend on whether K is actually knotted or not!]