Definition. We say that a function $f : \mathbf{R} \to \mathbf{R}$ is *additive* if, for all $x \in \mathbf{R}$ and all $y \in \mathbf{R}$,

$$f(x + y) = f(x) + f(y).$$

Lemma 1. Suppose $\phi : \mathbf{R} \to \mathbf{R}$ is additive and suppose ϕ is bounded on $\mathbf{R}$. Then $\phi(x) = 0$ for all $x \in \mathbf{R}$.

Proof. By contradiction. Suppose $\phi(x_0) \neq 0$ for some $x_0 \in \mathbf{R}$. Since ϕ is additive, it follows by induction that $\phi(nx_0) = n\phi(x_0)$ for all $n \in \mathbf{N}$. Since $\phi(x_0) \neq 0$, then $(n\phi(x_0))$ is an unbounded sequence, so $(\phi(nx_0))$ is unbounded. But this is impossible if ϕ is bounded.

Definition. Suppose $\phi : \mathbf{R} \to \mathbf{R}$, and suppose $T > 0$. We say ϕ is *periodic with period* T if, for all $x \in \mathbf{R}$,

$$\phi(x + T) = \phi(x).$$

Lemma 2. Suppose $\phi : \mathbf{R} \to \mathbf{R}$ is periodic with period T, and suppose ϕ is bounded on some closed interval of length T. Then ϕ is bounded on all of $\mathbf{R}$.

Proof. If ϕ is bounded on some interval I of length T, then there exists M such that $|\phi(x)| \leq M$ for all $x \in I$. To prove the theorem it suffices to show that $|\phi(y)| \leq M$ for all $y \in \mathbf{R}$.

Write I in the form $I = [a, a + T]$. For every real number y, there exists an integer n such that $y \in [a+nT, a+(n+1)T]$ (one way to prove this is to define n to be the largest integer such that $a+nT \leq y$). Therefore $y - nT \in [a, a + T] = I$, so $|\phi(y - nT)| \leq M$. But it is easy to see from the definition of periodic function with period T that $\phi(y - nT)$ must equal $\phi(y)$. So $|\phi(y)| \leq M$.

Theorem. Suppose $f : \mathbf{R} \to \mathbf{R}$ is additive and suppose f is bounded on some closed interval $[a, b]$, where $a < b$. Then there exists a number A such that $f(x) = Ax$ for all $x \in \mathbf{R}$.

Proof. Let $T = b - a$ be the length of the interval $[a, b]$. Define $A = f(T)/T$, and define

$$\phi(x) = f(x) - Ax$$

for all $x \in \mathbf{R}$. For all $x \in \mathbf{R}$ and all $y \in \mathbf{R}$ we have

$$\phi(x + y) = f(x + y) - A(x + y) = f(x) + f(y) - Ax - Ay = (f(x) - Ax) + (f(y) - Ay) = \phi(x) + \phi(y).$$

This proves that ϕ is additive. Moreover, since $f(x)$ and Ax are both bounded functions on $[a, b]$, then their difference $\phi(x)$ is also bounded on $[a, b]$.

On the other hand, notice that $\phi(T) = f(T) - AT = 0$, by definition of A. Hence, for all $x \in \mathbf{R}$ we have, since ϕ is additive,

$$\phi(x + T) = \phi(x) + \phi(T) = \phi(x) + 0 = \phi(x).$$

This shows that ϕ is periodic on $\mathbf{R}$, and since we already know it is bounded on an interval of length T, Lemma 2 tells us that ϕ is bounded on all of $\mathbf{R}$. Then Lemma 1 tells us that $\phi(x) = 0$ for all $x \in \mathbf{R}$. From the definition of ϕ it follows that $f(x) - Ax = 0$ for all $x \in \mathbf{R}$, which is what we wanted to prove.

(This proof was taken from the article *The linear functional equation* by G.S. Young, in *The American Mathematical Monthly*, Vol. 65 (1958), pp. 37–38.)